Eigenvalues of conical waveguides

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Based on collaborations with
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Thomas Ourmières-Bonafos

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  Example: Coulomb potential $V(x) = -1/|x|$ in $L^2(\mathbb{R}^3)$, the eigenvalues: $-\dfrac{1}{4n^2}$.

  Weyl asymptotics: the number of eigenvalues below $-\varepsilon$, $N(A, -\varepsilon)$, satisfies

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- 1D model: $T = -\dfrac{d^2}{dx^2} + V(x)$ in $L^2(\mathbb{R}_+)$ with $V \sim -\dfrac{c}{x^2}$ at ∞: $[0, \infty) \subset$ spectrum,

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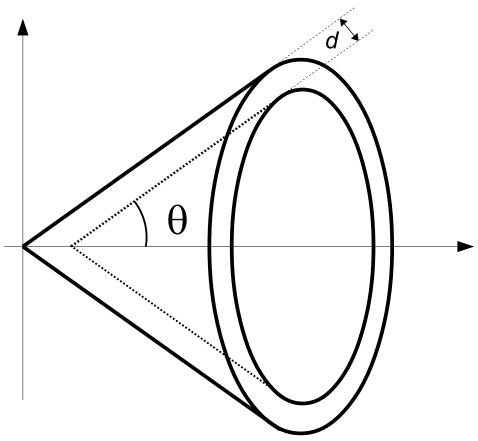
We are going to study the eigenvalues for several models involving “conical waveguides” whose cross-section grows linearly at infinity.

Previous works

Conical layer:



$\theta \neq \dfrac{\pi}{2}$: Exner and Tater (2010): Dirichlet Laplacian A in a rotationally invariant infinite conical layer of constant width d. Continuous spectrum = $\left[\dfrac{\pi^2}{d^2}, +\infty\right)$, infinitely many eigenvalues below $\dfrac{\pi^2}{d^2}$.

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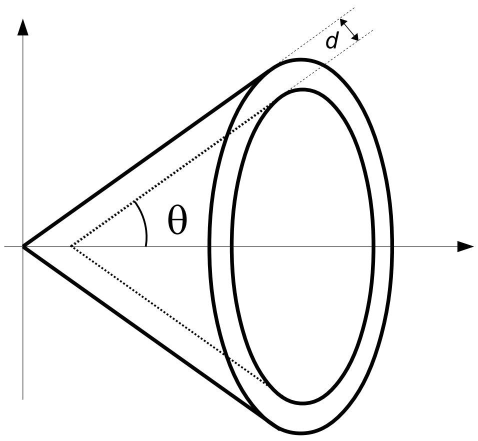

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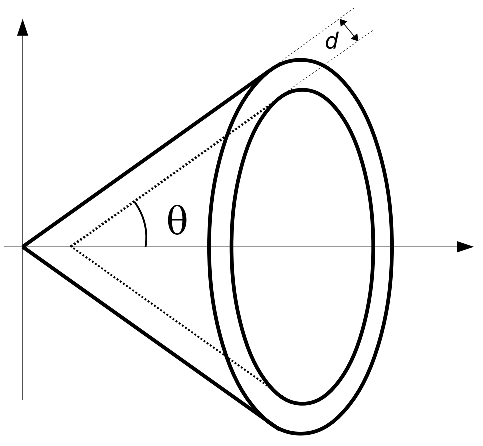

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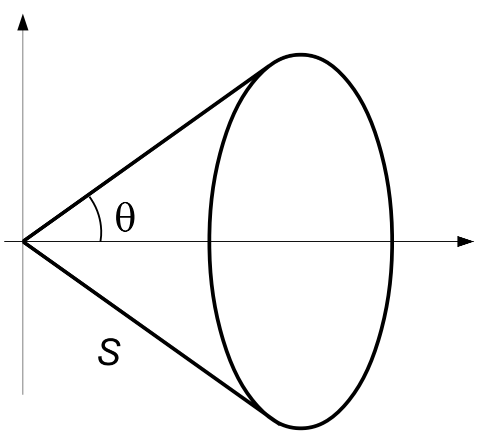

Let A be the Laplacian with the jump $[\partial u/\partial n] = \alpha u$ at S; the quadratic form is

$$H^1(\mathbb{R}^3) \ni u \mapsto \int_{\mathbb{R}^3} |\nabla u|^2 dx - \alpha \int_S |u|^2 ds, \quad \alpha > 0.$$

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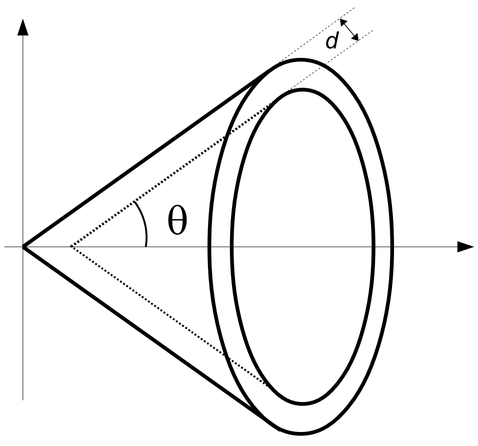

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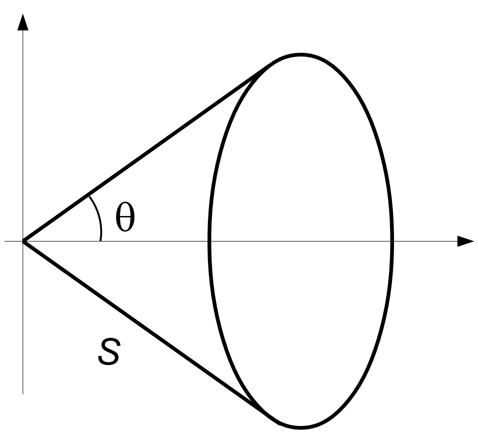

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Lotoreichik, Ourmières-Bonafos (preprint 2015): the same asymptotics for $N(A, -\alpha^2/4 - \varepsilon)$.

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We say that a surface/domain $X \subset \mathbb{R}^3$ is *conical* if

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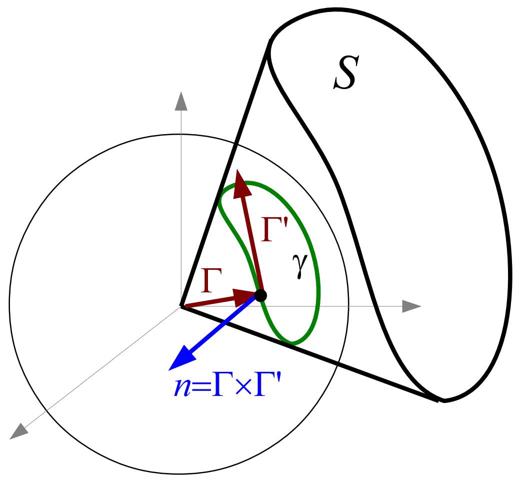
Three models:

- Ω is a conical layer, Dirichlet Laplacian in Ω,
- S is a conical surface, Schrödinger operator with a δ-potential on S,
- Ω is a conical domain, the Laplacian in Ω with Robin boundary condition

$$\frac{\partial u}{\partial n} = \alpha u.$$

Curvature and layers



S conical surface, **the cross-section** $\gamma \subset \mathbb{S}^2$ **is a simple** C^4 **loop** with an arc-length parametrization

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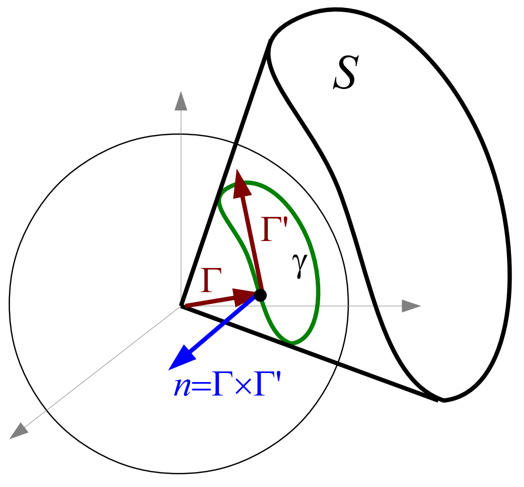

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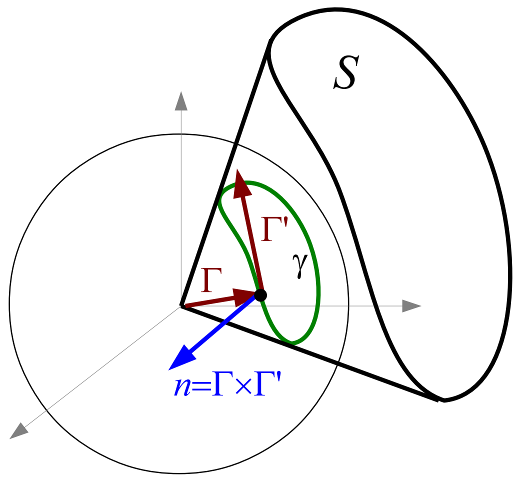

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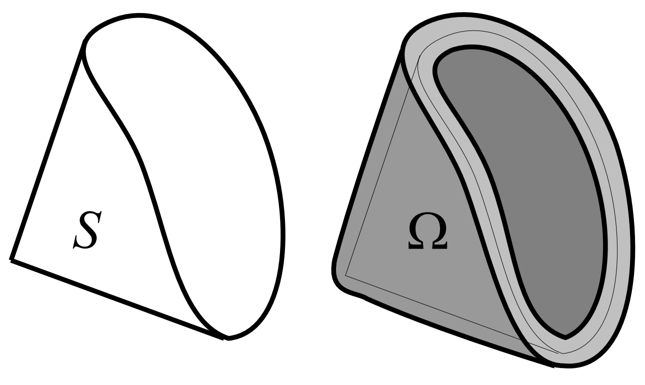

Theorem. Let Ω be a conical layer of width d around S and A be the Dirichlet Laplacian in Ω, then

$$N\left(A, \frac{\pi^2}{d^2} - \varepsilon\right) \sim C|\ln \varepsilon|, \quad \varepsilon \to 0, \quad C = \sum_{j=1}^{\infty} \sqrt{(-\mu_j)_+},$$

where μ_j are the eigenvalues of

$$-\frac{d^2}{ds^2} - \frac{\kappa^2}{4} \text{ on } \mathbb{T}.$$

If γ is not a big circle ($\sim S$ not a plane), then $C > 0$ and there are infinitely many eigenvalues.

Proof: Adapted coordinates

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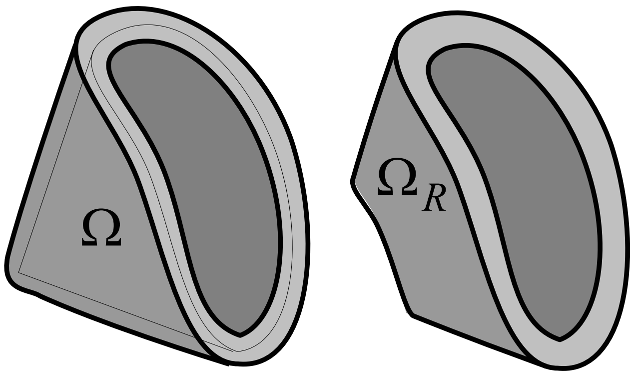
$$\Omega_R := X\Big((R,+\infty)\times\mathbb{T}\times\big(-\frac{d}{2},\frac{d}{2}\big)\Big), \quad X(r,s,t) = r\Gamma(s) + t\Gamma(s)\times\Gamma'(s).$$



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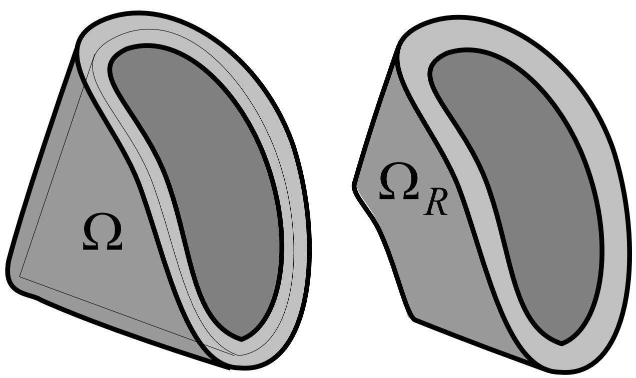

The Dirichlet Laplacian A_Ω in Ω satisfies

$$\text{continuous spectrum} = \Big[\frac{\pi^2}{d^2},+\infty\Big),$$

$$N\Big(A_{R,D},\frac{\pi^2}{d^2}-\varepsilon\Big) - C \le N\Big(A_\Omega,\frac{\pi^2}{d^2}-\varepsilon\Big) \le N\Big(A_{R,N},\frac{\pi^2}{d^2}-\varepsilon\Big) + C,$$

where $A_{R,D/N}$ are Laplacians in Ω_R, Dirichlet on $\partial\Omega\cap\partial\Omega_R$, Dirichlet/Neumann at $|x| = R$.

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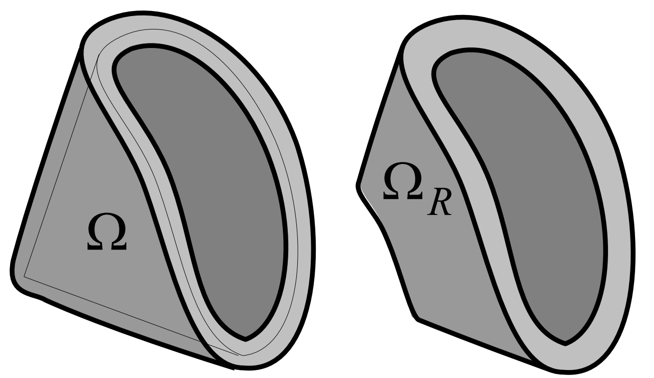

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The asymptotics of $N\Big(A_{R,D/N}, \dfrac{\pi^2}{d^2} - \varepsilon\Big)$ for $\varepsilon \to 0$?

Proof: 1D reduction

One passes to $L^2(\Pi, dr\,ds\,dt)$, $\Pi := (R,+\infty)\times\mathbb{T}\times\big(-\dfrac{d}{2},\dfrac{d}{2}\big)$.

After some transformations, one minorates/majorates the quadratic forms for $A := A_{R,N/D}$ in (r,s,t) by

$$\int_\Pi \left(v_r^2 + \frac{1}{r^2}\left(v_s^2 - \frac{\kappa^2+1}{4}v^2\right) + v_t^2 + \frac{c}{r^3}v^2\right) dr\,ds\,dt + c\int_{r=R} v^2 ds\,dt, \quad v(r,\cdot) = 0 \text{ for } D.$$

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Then $A\simeq\bigoplus_{j,k}A_{j,k}$ with $A_{j,k}=-\dfrac{d^2}{dr^2}+\dfrac{\mu_j-1/4}{r^2}+\dfrac{c}{r^3}+\Big(\dfrac{\pi k}{d}\Big)^2$ in $L^2(R,\infty)$ (+ b.c. at R).

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Only $k=1$ and $\mu_j < 0$ contribute, and one can use the known 1D result.

δ-interaction

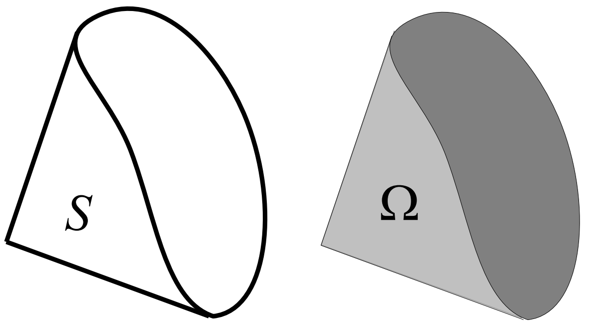
Under the same assumptions on S, literally the same result is expected for the Schrödinger operators with δ-potentials: the associated quadratic form is

$$H^1(\mathbb{R}^3) \ni u \mapsto \int_{\mathbb{R}^3} |\nabla u|^2 dx - \alpha \int_S |u|^2 ds.$$

Some technical details are still be be worked out: the analysis on a conical layer of a non-constant width around S (in preparation, with T. Ourmières-Bonafos)

Robin Laplacian

Let Ω be bounded by a conical surface S as above.



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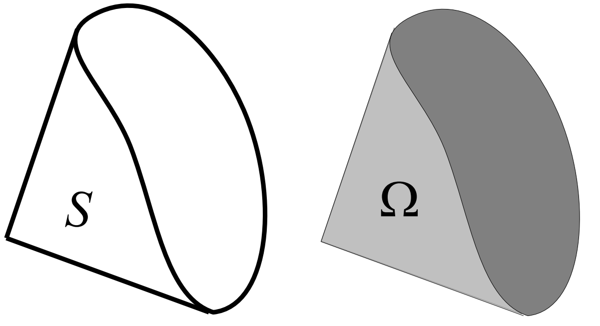

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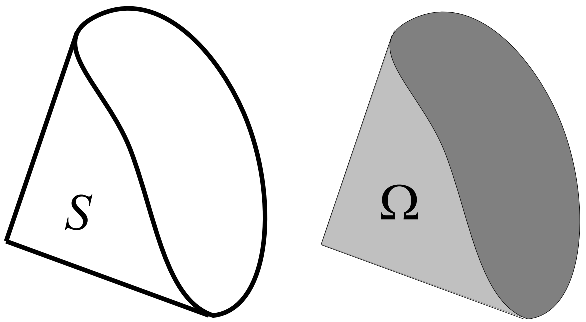

For $\alpha > 0$, consider Q^Ω_α Laplacian in $L^2(\Omega)$ with Robin boundary conditions:

$$u \mapsto -\Delta u, \quad \frac{\partial u}{\partial n} = \alpha u \text{ at } S.$$

The quadratic form is

$$H^1(\Omega) \ni u \mapsto \int_\Omega |\nabla u|^2 dx - \alpha \int_S |u|^2 ds.$$

Difference to the previous cases:

Theorem (KP'2015). The continuous spectrum is $[-\alpha^2, \infty)$ and:

- If Ω is the exterior of a convex set, then the discrete spectrum is empty.
- Otherwise, infinitele many eigenvalues below $-\alpha^2$.

(Attention: smoothness of the boundary is important!)

Proof for convex Ω^c

Continuous spectrum $[-\alpha^2, +\infty)$: standard (cutting out a compact part + approximate eigenfunctions): Bruneau, Popoff (2016).

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$$\begin{aligned}
&\int_\Omega |\nabla u|^2 ds - \alpha \int_S |u|^2 ds + \alpha^2 \int_\Omega |u|^2 dx \\
&\quad \geq \int_{\Phi(S\times\mathbb{R}_+)} |\nabla u|^2 ds - \alpha \int_S |u|^2 ds + \alpha^2 \int_{\Phi(S\times\mathbb{R}_+)} |u|^2 dx \\
&\quad\quad = \int_{S\times\mathbb{R}_+} \Big|(\nabla u)\big(\Phi(s,t)\big)\Big|^2 J(s,t)\, ds\, dt - \alpha \int_S |u|^2 ds + \alpha^2 \int_{S\times\mathbb{R}_+} \Big|u\big(P(s,t)\big)\Big|^2 J(s,t)\, ds\, dt =: I.
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Due to $k_j\leq 0$ we have $J\geq 1$, and

$$\Big|(\nabla u)\big(\Phi(s,t)\big)\Big|^2\geq\Big|(n\cdot\nabla u)\big(\Phi(s,t)\big)\Big|^2=\Big|\frac{\partial}{\partial t}u\big(\Phi(s,t)\big)\Big|^2,$$

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and

$$I\geq\int_S\bigg(\int_{\mathbb{R}_+}\Big|\frac{\partial}{\partial t}u\big(\Phi(s,t)\big)\Big|^2 dt-\alpha\big|u(s,0)\big|^2+\alpha^2\int_{\mathbb{R}_+}\Big|u\big(\Phi(s,t)\big)\Big|^2 dt\bigg)ds\geq 0,$$

i.e. $Q_\alpha^\Omega > -\alpha^2$.

Counting the eigenvalues

Theorem (Bruneau, KP, Popoff, 2016).

$$N(Q_\alpha^\Omega, -\alpha^2 - \varepsilon) \sim \frac{\alpha^2}{8\pi\varepsilon} \int_{\mathbb{T}} (\kappa_+)^2\, ds, \quad \varepsilon \to 0. \qquad (\star)$$

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$$E_j(Q_\alpha^\Omega) \sim -\alpha^2 + E_j(-\Delta_S - \alpha H) + O(1) \text{ as } \alpha \to +\infty,$$

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But in our case α is fixed!

Proof: Polar coordinates

Let $\Sigma := \Omega \cap \{|x| = 1\}$ be the cross section (spherical domain), then

$$\mathbb{R}_+ \times \Sigma \ni (r, \theta) \mapsto r\theta$$

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Theorem. For some $C > 0$, there holds

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Weyl asymptotics for $N(L_j, -\varepsilon)$: manual proof.