Halfspace Matching for 2D Open Waveguides

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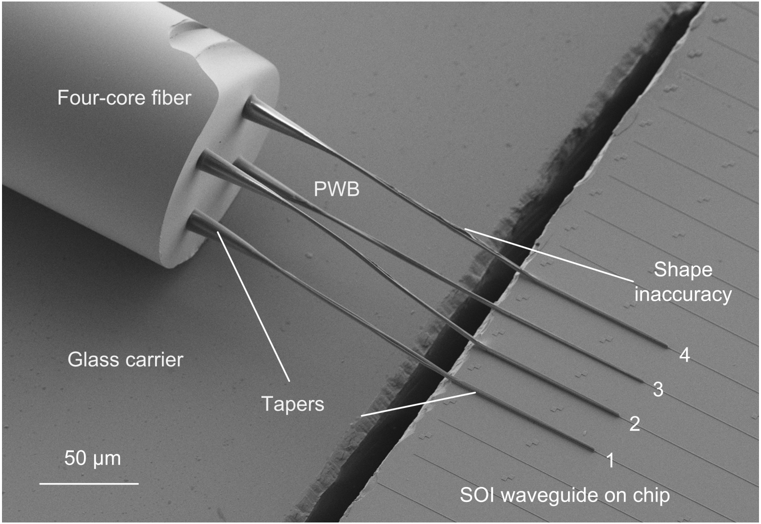
Joint work with A.S. Bonnet-Ben Dhia, S. Fliss, C. Hazard, A. Tonnoir

19th of May 2016, Waveguides 2016, Porquerolles



Lindenmann, N.; Dottermusch, S.; Goedecke, M. L.; Hoose, T.; Billah, M. R.; Onanuga, T.; Hofmann, A.; Freude, W.; Koos, C. **Connecting Silicon Photonic Circuits to Multi-Core Fibers by Photonic Wire Bonding** *J. Lightwave Technol.* 33, 755 - 760 (2014)

Overview

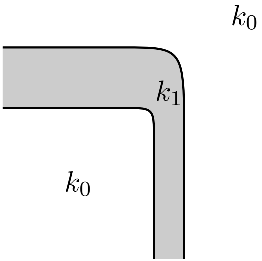
1. **Halfspace Matching for Waveguides**

2. Numerical Examples

A very "simple" model

Given $k_1 > k_0 > 0$ and some incident mode U_{inc}, find $U : \mathbb{R}^2 \to \mathbb{C}$ such that

$$\begin{cases} \Delta U + k^2(x)U = 0 \quad \text{in } \mathbb{R}^2 \\ U - U_{inc} \text{ fulfills some radiation condition} \end{cases}$$



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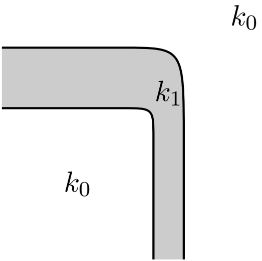


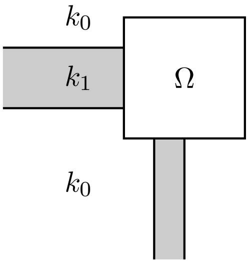
Questions:

- (?) Framework, radiation condition, well-posedness
- (?) Numerical treatment

An even simpler model

Given Dirichlet data g on $\partial\Omega$, find $U : \mathbb{R}^2 \setminus \overline{\Omega} \to \mathbb{C}$ such that

$$\begin{cases} \Delta U + (k^2(x) + i\epsilon)U = 0 & \text{in } \mathbb{R}^2 \setminus \Omega \\ U = g & \text{on } \partial\Omega \\ U \in H^1(\mathbb{R}^2 \setminus \Omega) \end{cases}$$



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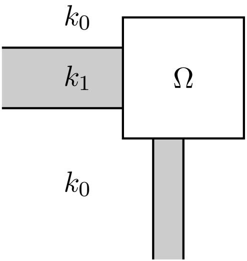

Questions:

- Framework, well-posedness: Clear! (H^1 framework, Lax Milgram)
- (?) Numerical treatment

Numerical Methods

PML

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Hardy space infinite elements — Hohage, Nannen, ...

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Local absorbing boundary conditions	Klindworth, Schmidt

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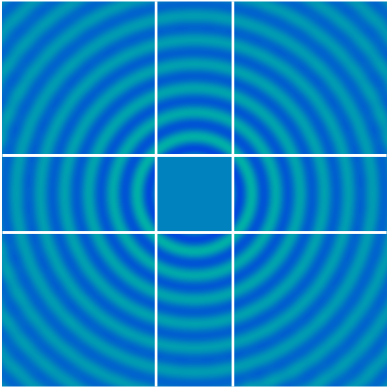
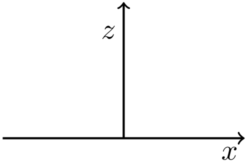


Plane wave expansion

Let $k_0, \epsilon > 0$ and let u be the solution of

$$\Delta u + (k_0^2 + i\epsilon)u = 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}_+$$
$$u(x,0) = u^0(x) \quad x \in \mathbb{R}$$



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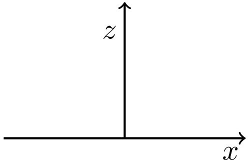

[Idea: $\Delta + k_0^2 = \partial_z^2 + \underbrace{\partial_x^2 + k_0^2}_{\text{diagonalized by } \mathcal{F}}$].

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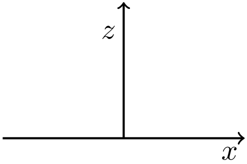

Then u can be expanded in the Form

$$u(x,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\sqrt{\xi^2 - k_0^2 - i\varepsilon}\, z} e^{-i\xi x} \langle u^0, e^{-i\xi \cdot} \rangle \, d\xi.$$

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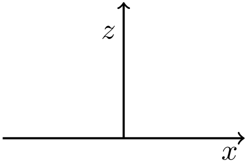

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$$u = \mathcal{S}_{\text{free}} u^0$$

[Idea: $\Delta + k_0^2 = \partial_z^2 + \underbrace{\partial_x^2 + k_0^2}_{\text{diagonalized by } \mathcal{F}}$].

Modal expansion for the waveguide

We have to find u such that

$$\Delta u + (k(x)^2 + i\epsilon)u = 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}_+$$
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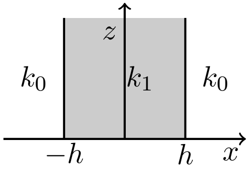
where $k_1, k_0, h > 0$ are constant.



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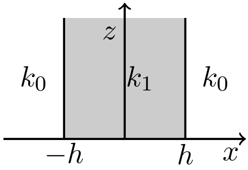

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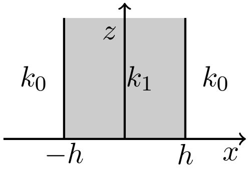

where $k_1, k_0, h > 0$ are constant.

The solution is given by

$$u(x,z) = \sum_{n=1}^{N} e^{-\sqrt{\xi_n^2 - k_0^2 - i\varepsilon}\, z}\langle u^0, \phi_n\rangle \phi_n(x)$$
$$+ \sum_{\nu\in\{s,a\}} \int_0^\infty e^{-\sqrt{\xi^2 - k_0^2 - i\varepsilon}\, z}\langle u^0, \phi_\xi^\nu\rangle \phi_\xi^\nu(\xi)\, \mu_\nu(\xi) d\xi, \quad x \in \mathbb{R}$$

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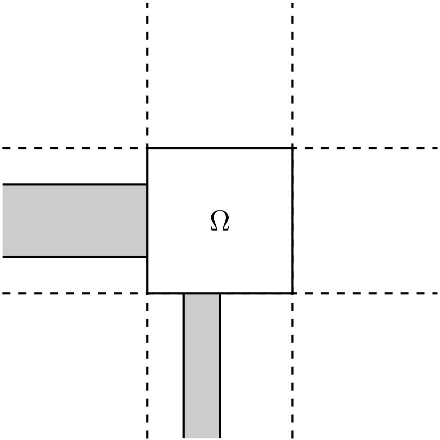
The System of Integral equations

Let $U : \mathbb{R}^2 \setminus \Omega \to \mathbb{C}$ be the solution with Dirichlet data g.



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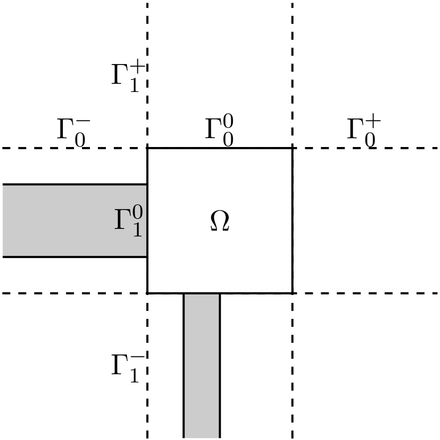
Let $U : \mathbb{R}^2 \setminus \Omega \to \mathbb{C}$ be the solution with Dirichlet data g. Define:

$$u_n^{\pm} = U|_{\Gamma_n^{\pm}}, \quad n \in \{0, \ldots, 3\}$$



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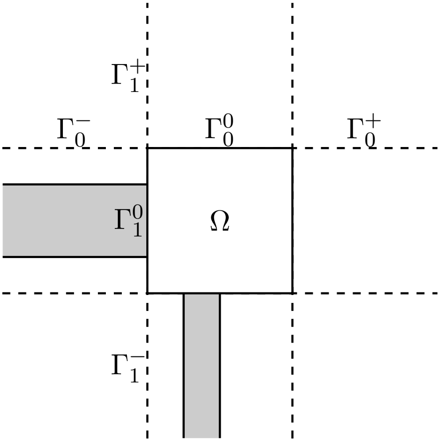

Compatibility equations

$$u_{n\pm1}^{\pm} = (\mathcal{S}_n(u_n^+ + g_n + u_n^-))|_{\Gamma_{n\pm1}^{\pm}}$$

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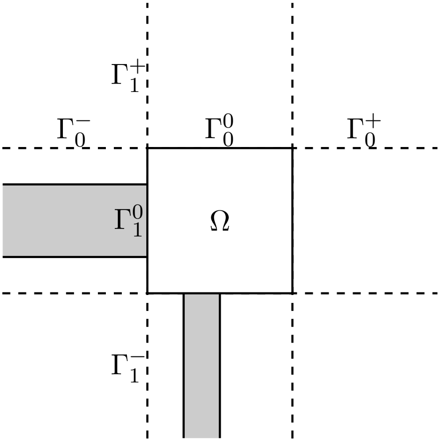

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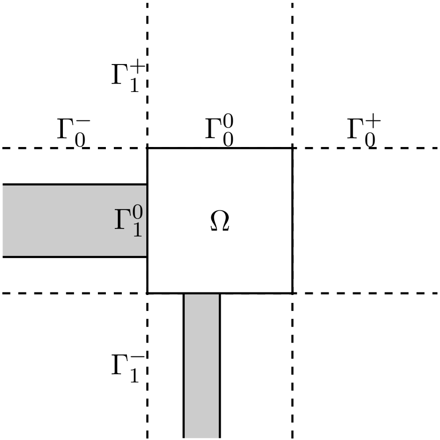
We have to consider the operators

$$\mathcal{S}_n|_{\Gamma_{n\pm1}^\pm} : H^1(\Gamma_n) \to H^1(\Gamma_{n\pm1}^\pm)$$



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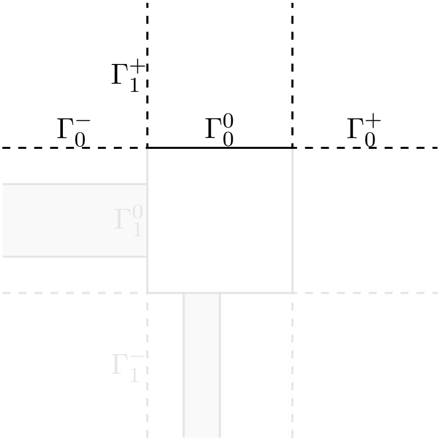

A. Tonnoir, *Conditions transparentes pour la diffraction d'ondes en milieu élastique anisotrope*, PhD Thesis, 2015, ENSTA ParisTech

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- One can show that

$$\mathcal{S}_0|_{\Gamma_1^+} : H^1(\Gamma_0^+) \to H^1(\Gamma_1^+)$$

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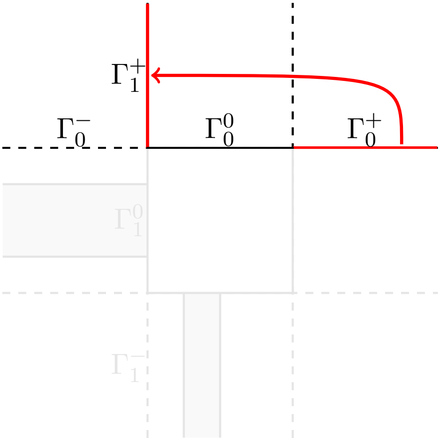

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is compact.

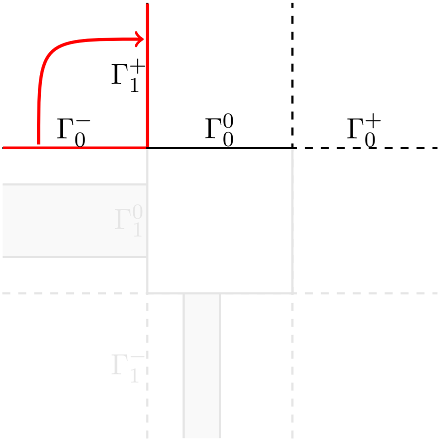
- The Operator

$$\mathcal{S}_0|_{\Gamma_1^+} : H^1(\Gamma_0^-) \to H^1(\Gamma_1^+)$$

can be split

$$\mathcal{S}_0|_{\Gamma_1^+} = C + B$$

C is compact, $\|B\| < 1$.



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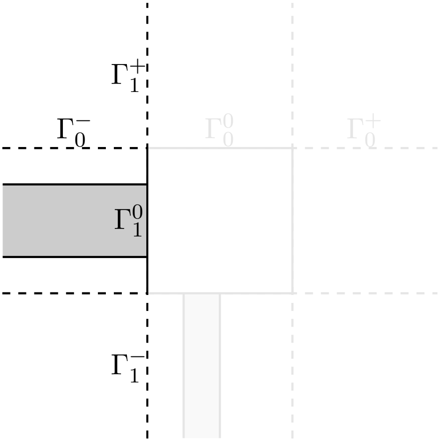


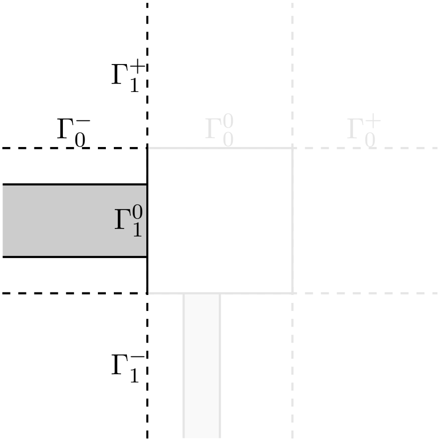
Mapping Properties – for the waveguide

The operator

$$(\mathcal{S}_{\text{waveguide}} - \mathcal{S}_{\text{free}})|_\Gamma$$

is compact.

⇒ The mapping properties carry over to the waveguide case.



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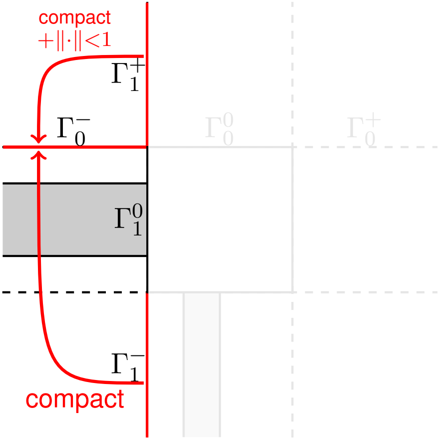

The full System

$$u_{m\pm1}^{\pm} - \underbrace{\mathcal{S}_m|_{\Gamma_{m\pm1}^{\pm}}}_{=:\mathcal{S}_m^{\pm}}(u_m^+ + u_m^-) = \mathcal{S}_m|_{\Gamma_{m\pm1}^{\pm}} g_n \qquad m \in \{0, \dots, 3\}, \pm$$

The full System

$$u^{\pm}_{m\pm1} - \underbrace{\mathcal{S}_m|_{\Gamma^{\pm}_{m\pm1}}}_{=:\mathcal{S}^{\pm}_m}(u^+_m + u^-_m) = \mathcal{S}_m|_{\Gamma^{\pm}_{m\pm1}} g_n \qquad m \in \{0,\dots,3\}, \pm$$

As a Matrix:

$$I - \left[\begin{array}{cc|cc|cc|cc}
 & & & & & & \mathcal{S}_3^+ & \mathcal{S}_3^+ \\
 & & \mathcal{S}_1^- & \mathcal{S}_1^- & & & & \\
\hline
\mathcal{S}_0^+ & \mathcal{S}_0^+ & & & & & & \\
 & & & & \mathcal{S}_2^- & \mathcal{S}_2^- & & \\
\hline
 & & \mathcal{S}_1^+ & \mathcal{S}_1^+ & & & & \\
 & & & & & & \mathcal{S}_3^- & \mathcal{S}_3^- \\
\hline
 & & & & \mathcal{S}_2^+ & \mathcal{S}_2^+ & & \\
\mathcal{S}_0^- & \mathcal{S}_0^- & & & & & &
\end{array}\right]
\left(\begin{array}{c}
u_0^+ \\ u_0^- \\ \hline u_1^+ \\ u_1^- \\ \hline u_2^+ \\ u_2^- \\ \hline u_3^+ \\ u_3^-
\end{array}\right) = F$$

The full System

$$u^{\pm}_{m\pm 1} - \underbrace{\mathcal{S}_m|_{\Gamma^{\pm}_{m\pm 1}}}_{=:\mathcal{S}^{\pm}_m}(u^+_m + u^-_m) = \mathcal{S}_m|_{\Gamma^{\pm}_{m\pm 1}} g_n \qquad m \in \{0,\dots,3\}, \pm$$

As a Matrix:

$$I - \left[\begin{array}{cc|cc|cc|cc}
 & & & & & & \color{blue}{\mathcal{S}^+_3} & \color{red}{\mathcal{S}^+_3} \\
 & & \color{red}{\mathcal{S}^-_1} & \color{blue}{\mathcal{S}^-_1} & & & & \\
\hline
\color{blue}{\mathcal{S}^+_0} & \color{red}{\mathcal{S}^+_0} & & & & & & \\
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\begin{pmatrix} u^+_0 \\ u^-_0 \\ \hline u^+_1 \\ u^-_1 \\ \hline u^+_2 \\ u^-_2 \\ \hline u^+_3 \\ u^-_3 \end{pmatrix} = F$$

compact + ($\|\cdot\| < 1$)

compact

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Only non compact operators:

$$\mathcal{B} = I - \left[\begin{array}{cc|cc|cc|cc}
 & & & & & & & B_3^+ \\
 & & B_1^- & & & & & \\ \hline
 & B_0^+ & & & & & & \\
 & & & & B_2^- & & & \\ \hline
 & & & B_1^+ & & & & \\
 & & & & & & B_3^- & \\ \hline
 & & & & & B_2^+ & & \\
B_0^- & & & & & & &
\end{array}\right]$$

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$\Rightarrow$ $\mathcal{B}$ is boundedly invertable.

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Can be written as

$$(\mathcal{B} + \mathcal{C})\mathcal{U} = F,$$

where $\mathcal{B}$ is boundedly invertable, $\mathcal{C}$ is compact.

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- Kern$(\mathcal{B} + \mathcal{C}) = \{0\}$ (by uniqueness of Helmholtz equation)

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$\Rightarrow$ Problem is usable for FE – discretization.

Further Remarks

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- Other formulations ensuring stability with FEM: $\rightarrow$ A. Tonnoirs Thesis

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- Other formulations ensuring stability with FEM: $\rightarrow$ A. Tonnoirs Thesis
- Non-absorptive case ($\epsilon = 0$): Functional Framework?

Overview

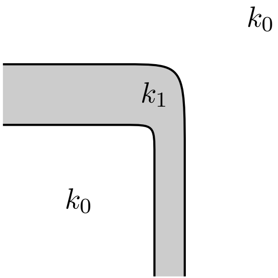
1. Halfspace Matching for Waveguides

2. **Numerical Examples**

Coupling with FEM

Given $k_1 > k_0 > 0$ and some incident mode U_{inc}, find $U : \mathbb{R}^2 \to \mathbb{C}$ such that

$$\begin{cases} \Delta U + k^2(x)U = 0 \quad \text{in } \mathbb{R}^2 \\ U - U_{inc} \text{ fulfills some rad. cond.} \end{cases}$$



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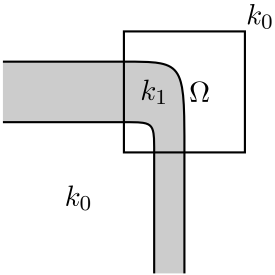

Setting $w = U|_{\Omega}, \; u = U|_{\mathbb{R}^2 \setminus \overline{\Omega}}$

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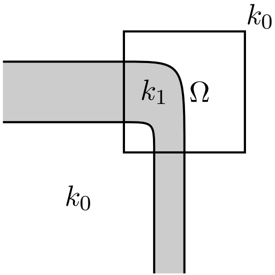
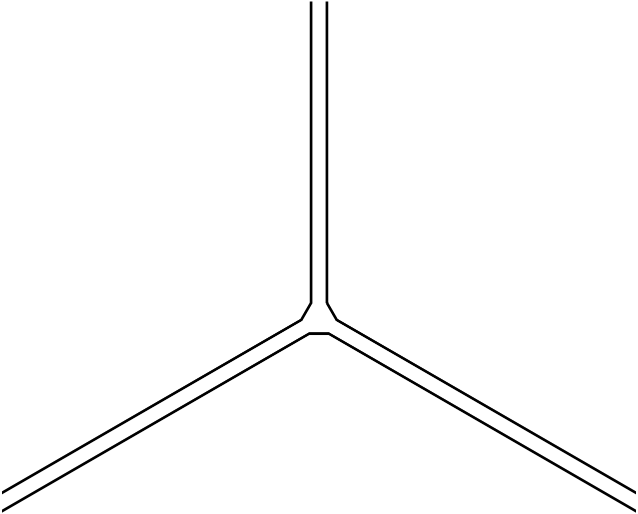
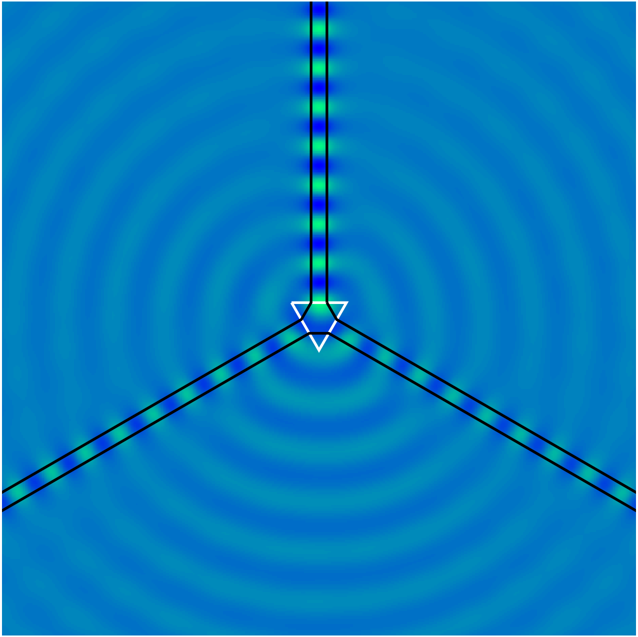
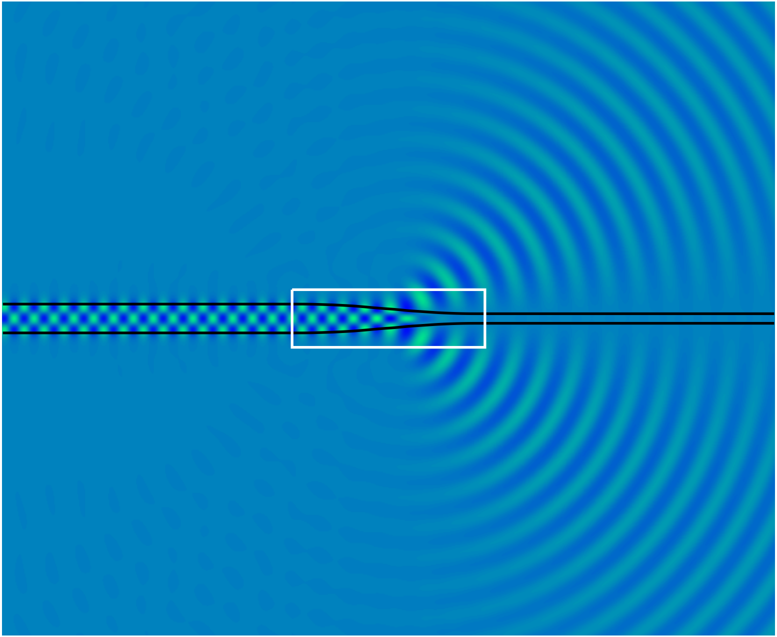


Setting $w = U|_{\Omega},\ u = U|_{\mathbb{R}^2\setminus\overline{\Omega}}$, this can reformulated into

$$\begin{aligned} \Delta w + k^2(x) w &= 0 && \text{in } \Omega \\ w - u &= 0 && \text{on } \partial\Omega \\ \partial_\nu w - \partial_\nu u &= f_{inc} && \text{on } \partial\Omega \\ \mathcal{S}_{n\pm1}|_{\Gamma_n^\pm} u|_{\Gamma_{n\pm1}} - u|_{\Gamma_n^\pm} &= f_{inc} && \text{on } \Gamma_n^\pm. \end{aligned}$$



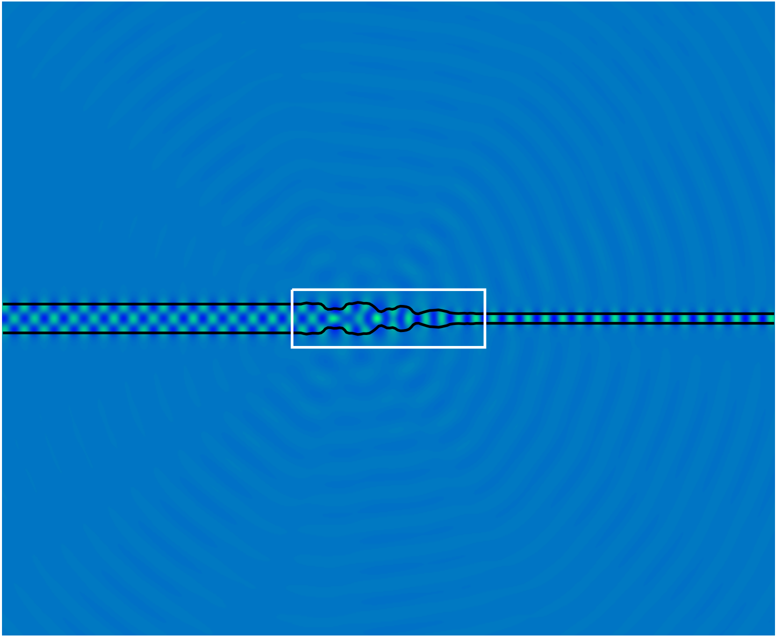




On Optimization

Use standard toolbox for shape optimization

- Domain derivatives
- Adjoint state
- Regularisation via surface metrics
- Descent algorithms (Quasi-Newton/CG)
- Deformation of the mesh via solutions of elasticity equations



The Last Slide

Methods works, but:

- No framework for the case without absorption
- Slow convergence (on test cases) wrt. truncation of the semi–infinite lines
- Geometrical limitations

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