Coupling of geometrical transformations and modal methods for waveguides of complex shape

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Outline

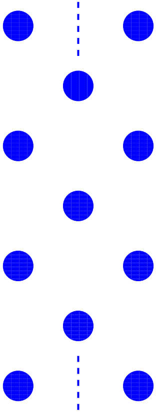
- Introduction
    - Initial motivation
    - General goal
- Geometrical transformations
- Simple case: straight guide with varying cross section
- Waveguide with circular restrictions
    - Parameters of the geometrical transformation
    - Case of several circles
- The numerical method
- The case of a half-waveguide
- Conclusion

Initial motivation

Determine waves guided by infinite arrays of penetrable scatterers

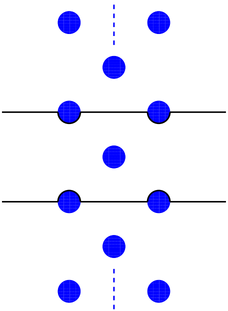
For one row:
- Neumann $\rightarrow$ guided waves can exist
- Dirichlet $\rightarrow$ no guided waves

A.-S. Bonnet-BenDhia *et al.*, Math. Mod. Meth. Appl. S. **17** (1994).

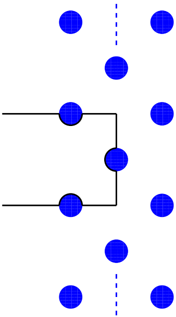


Reduction to a waveguide problem

Infinite waveguide



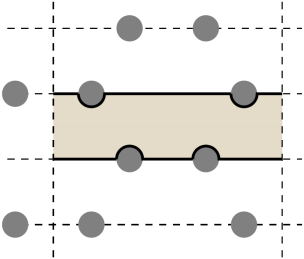
half-waveguide



Requires to be able to consider waveguides with:
- varying cross section,
- varying ending wall.

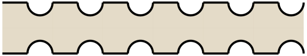
Other geometries

Propagation through an hexagonal gaz



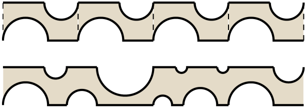
C. Perrot *et al.*, J. Acoust. Soc. Am. **124** (2008).

Corrugated waveguides



D. Tonon *et al.*, J. Sound Vib. **329** (2010).

Periodic and disordered Lorentz channels



T. Dittrich *et al.*, Chaos, Solitons and Fractals **8** (1997).

Outline

General goal

- Aim: develop modal methods, adapted to complex-shaped waveguides
- Method: use geometrical transformations to get a straight guide

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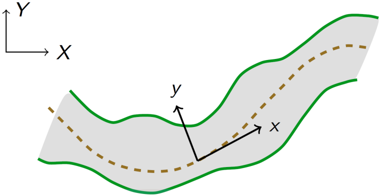
Geometrical transformations

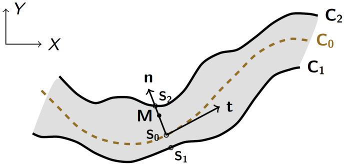
General complex-shaped waveguide considered:



Simple geometry after transformation:

Parametrisation

Let's consider $\boldsymbol{OM}(X, Y) = X\boldsymbol{i} + Y\boldsymbol{j}$. We can define $\boldsymbol{OM}(x, y)$ with:



$$\begin{aligned} x &= \text{arc length along } C_0, \\ \boldsymbol{OS}_{1,2}(x) &= \boldsymbol{OS}_0(x) + a_{1,2}(x)\boldsymbol{n}(x), \\ \boldsymbol{OM}(x, y) &= \boldsymbol{OS}_0(x) + b(x, y)\boldsymbol{n}(x), \end{aligned}$$

where

$$\begin{aligned} b(x, y) &= a_1(x) + y\, a(x), \\ a(x) &= a_2(x) - a_1(x), \end{aligned}$$

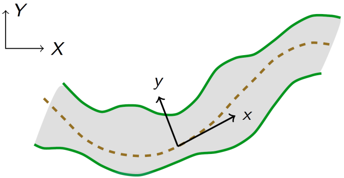
Valid for $\kappa(x)b(x, y) < 1$ where κ is the local curvature of C_0:

$$\frac{d\boldsymbol{OS}_0}{dx} = \boldsymbol{t} \quad \text{and} \quad \frac{d\boldsymbol{n}}{dx} = -\kappa(x)\boldsymbol{t}$$

Change of variables

$$(X, Y) \to (x, y)$$

$$\boldsymbol{OM} = X\boldsymbol{i} + Y\boldsymbol{j} = \boldsymbol{OS}_0(x) + b(x, y)\boldsymbol{n}(x)$$



$P(X, Y)$ solution of

$$\begin{aligned} (\Delta + k^2)P &= 0, \\ \boldsymbol{\nabla} P \cdot \boldsymbol{N} &= 0 \quad \text{on} \quad C_{1,2} \end{aligned}$$

$p(x, y)$ solution of

$$\begin{aligned} \operatorname{div}(H\boldsymbol{\nabla} p) + k^2 hp &= 0, \\ (H\boldsymbol{\nabla} p)\cdot \boldsymbol{e}_y &= 0 \quad \text{at} \quad y = 0, 1 \end{aligned}$$

Wave equation after geometrical transformation

$$\begin{cases} \operatorname{div}(H\boldsymbol{\nabla} p) + k^2 hp & = & 0, \\ (H\boldsymbol{\nabla} p)\cdot \boldsymbol{e}_y & = & 0 \quad \text{at} \quad y = 0,1 \end{cases}$$

with

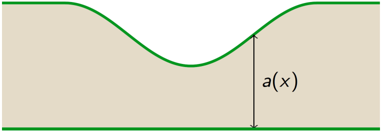
$$H = \frac{J^t J}{\det J} = \frac{1}{\tau}\begin{pmatrix} a & -\dfrac{\partial b}{\partial x} \\ -\dfrac{\partial b}{\partial x} & \dfrac{\tau^2 + (\partial b/\partial x)^2}{a} \end{pmatrix}$$

where

$$\begin{aligned} J & = \text{Jacobian of the transformation } (X,Y) \to (x,y), \\ \tau(x,y) & = 1 - \kappa(x) b(x,y), \\ h & = \frac{1}{\det J} = \tau a \end{aligned}$$

Outline

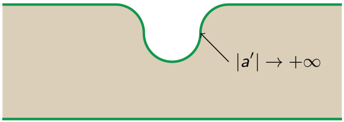
Simple case: straight guide with varying cross section



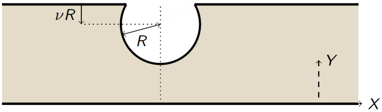
Change of variables:

$$x = X, \quad y = \frac{Y}{a(X)}$$

Limits of the simple change of variables



Waveguide containing restrictions with circular arc shape:



$\hookrightarrow$ half-circle impossible

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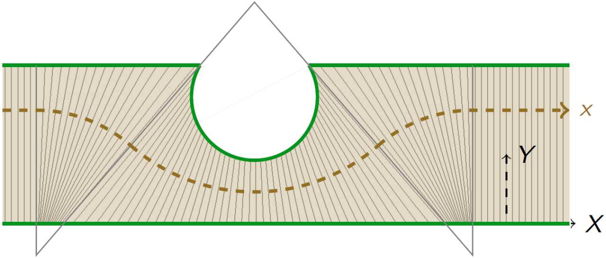
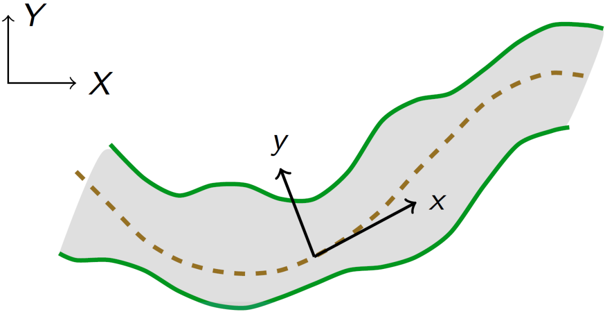
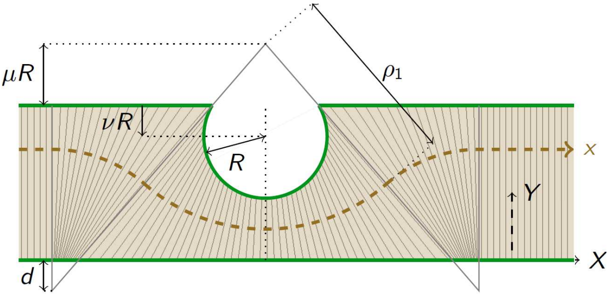
Good parametrisation



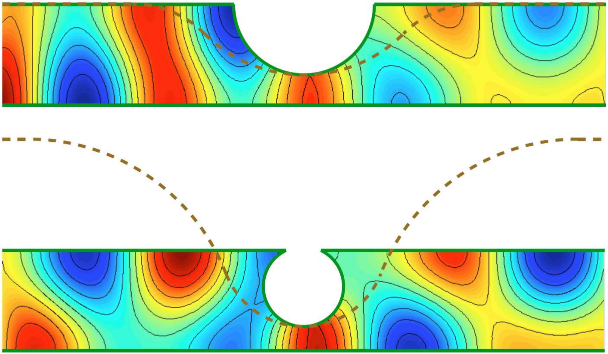
seen as:



Parameters of the geometrical transformation



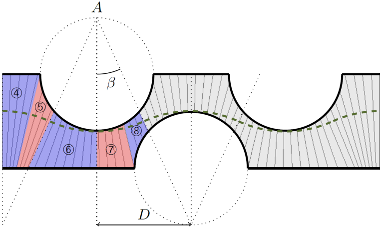
Examples



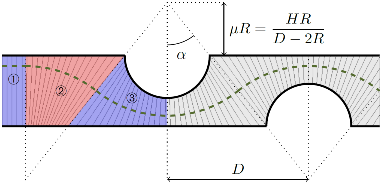
Outline

Case of several circles

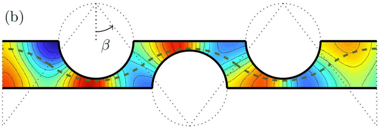
Transformation 2



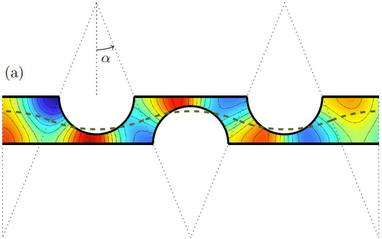
Transformation 1



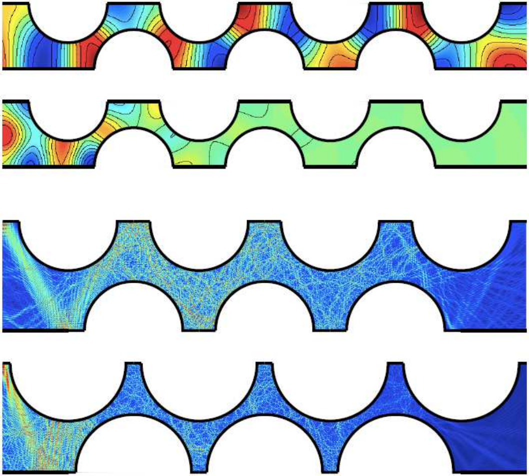
(b)



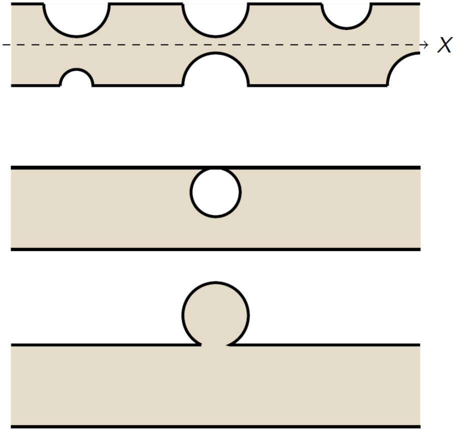
(a)



Some other numerical results



Three nontrivial extensions



Outline

The numerical method

Recall:

$$\begin{aligned} \operatorname{div}(H\boldsymbol{\nabla} p) + k^2 h p &= 0, \\ (H\boldsymbol{\nabla} p)\cdot \boldsymbol{e}_y &= 0 \quad \text{at} \quad y = 0, 1 \end{aligned}$$

Choices of modal decomposition

• Simple approach: $p^{(N)}(x,y) = \sum_{n=0}^{N-1} p_n(x)\varphi_n(y)$ where $\varphi_n = \cos(n\pi y)$ is the Neumann orthogonal basis of the transverse guide
$\hookrightarrow$ slow convergence

• Improved approaches:

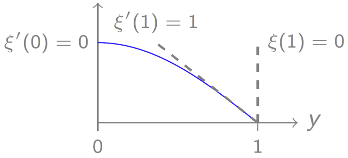
◦ For Robin condition:

$$\frac{\partial p}{\partial y}(x,0) = 0 \quad \text{and} \quad \frac{\partial p}{\partial y} = \eta(x)p \quad \text{at} \quad y = 1$$

$$p^{(N)}(x,y) = \sum_{n=0}^{N-1} p_n(x)\left[\varphi_n(y) + \eta(x)\varphi_n(1)\xi(y)\right]$$

W. Bi, V. Pagneux, D. Lafarge and Y. Aurégan, JASA **122** (2007)

with $\xi'(1) = 1$ and $\xi(1) = 0$
($\xi = \cos(\pi y/2)$)



Choices of modal decomposition

∘ For general conditions:

$$\frac{\partial p}{\partial y} = \eta(x)\frac{\partial p}{\partial x} \quad \text{at} \quad y = 1$$

Idea: extra unknown $\tilde{p}(x)$

$$p^{(N)}(x, y) = \sum_{n=0}^{N-2} p_n(x)\varphi_n(y) + \tilde{p}(x)\xi(y).$$

Initially developed for water waves

Athanassoulis *et al.* J. Fluid Mech. **389** (1999)

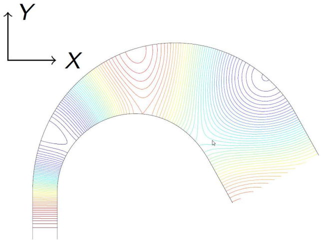
Convergence:

$$||p^{(\infty)} - p^{(N)}||_{H^1} \sim \frac{1}{N^{0.5}} \to \frac{1}{N^{2.5}}$$

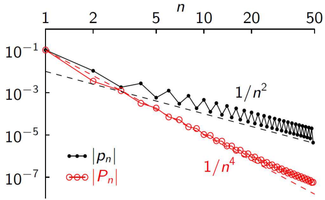
C. Hazard *et al.* IMA Journal of Applied Math. **73** (2008)

Convergence

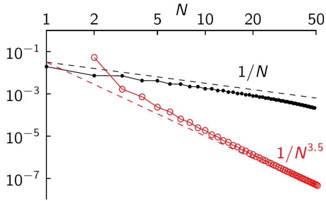
Elephant trunk



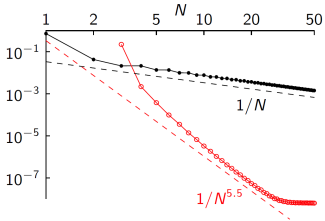
$||p_n||_{L^2}$



$||p^{(\infty)} - p^{(N)}||_{L^2}$

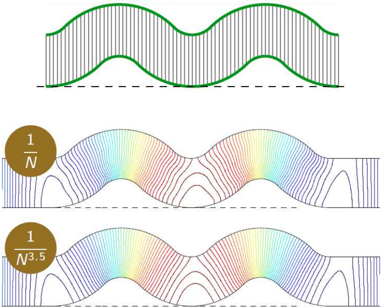


Refl. coef. (propagative modes)

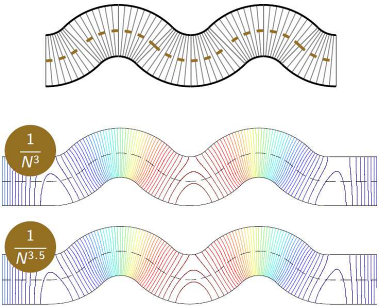


A funny configuration: the snake guide

Vertical change of variables



Orthogonal change of variables



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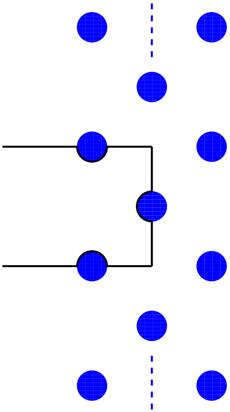
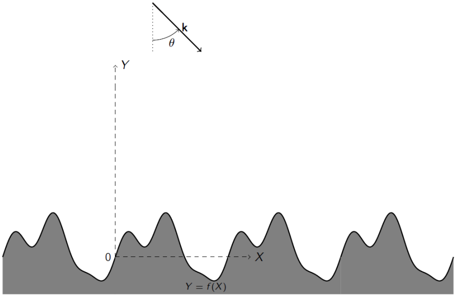
The case of a half-waveguide

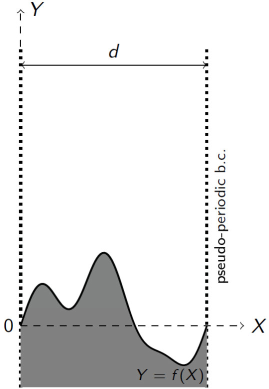


Scattering of a wave by an impenetrable grating

Infinite geometry



Reduction to a guide



Existing methods

• Rayleigh hypothesis: modal method valid everywhere
Rayleigh, Proc. R. Soc. Lond. A **79** (1907)
• Geometrical transformation, to get a flat surface
◦ C-method
- translation coordinate system which transforms the grating surface into a plane surface
- geometrical transformation that affects the whole space

J. Chandezon *et al.*, J. Opt. **11** (1980)

L. Li *et al.*, Pure Appl. Opt. **5** (1996)

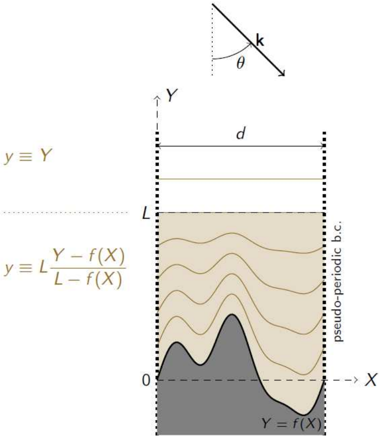
◦ local geometrical transformation
transformation in some bounded region containing the grating

A. A. Shcherbakov *et al.*, Opt. Expr. **21** (2013)

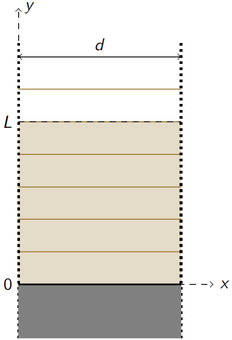
G. Favraud, and V. Pagneux, Proc. R. Soc. A **471** (2015)

Our method

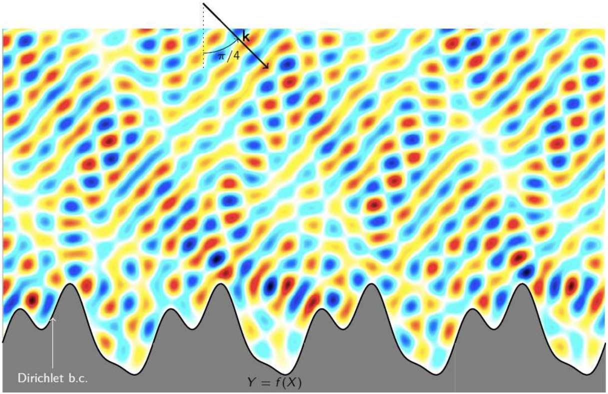
Before transformation



After transformation

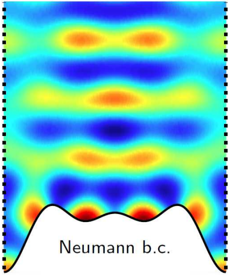


Example of scattering of a wave by an impenetrable grating

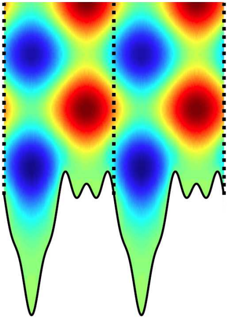


Other examples

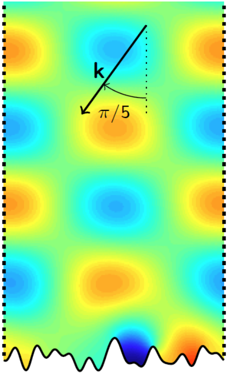
Neumann



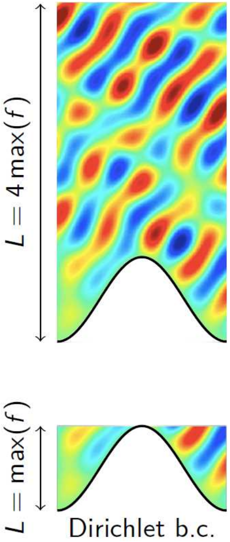
Dirichlet



Neumann random surface



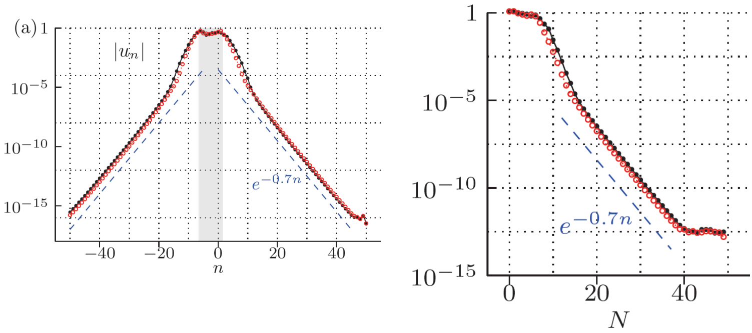
Influence of L



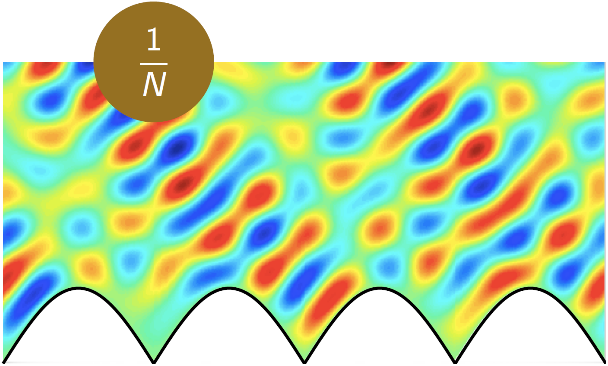
Convergence

Case $f \in C^\infty(\mathbb{R})$: exponential convergence

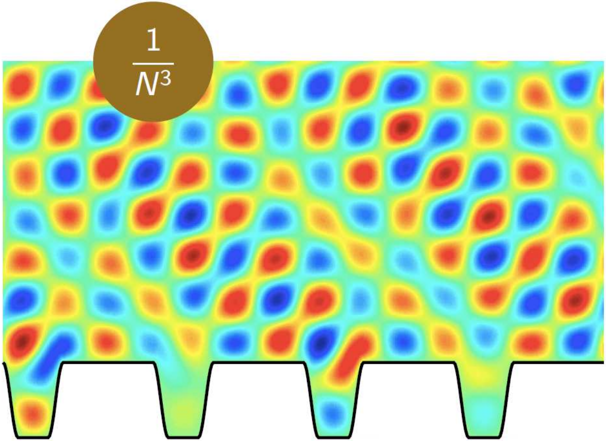
$||u^{(\infty)} - u^{(N)}||_{L^2}$



Profiles with discontinuous derivatives



Profiles with continuous derivatives



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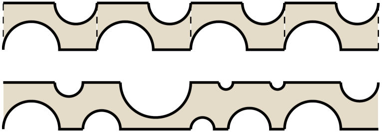
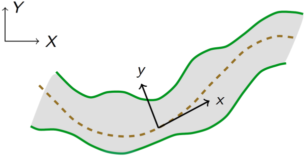
Initial geometry:

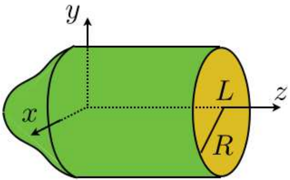


Final geometry:

Intermediate geometry:



Perspective: axisymmetric case



Thank you for your attention