Inverse boundary value problems by partial Cauchy data for Maxwell's equations and Schrödinger equations: cases of waveguides and cylindrical domains

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NEW TRENDS IN THEORETICAL AND NUMERICAL ANALYSIS OF WAVEGUIDES

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Contents

Hidden motivation: to be honest

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4. 2D Maxwell: maybe too simple!?

5. Well, let us consider

   3D Maxwell in waveguide: Uniqueness by partial DN map

Technical highlights

- Diagonalization of Maxwell's equations in waveguide to two 2D ellitpic equations
- Construction of complex geometric optics solutions

  cf. Imanuvilov-Uhlmann-Yamamoto (2010)

§1. Calderón Problem for Maxwell's equations

$\widetilde{\Omega} := \Omega \times (-\infty, \infty)$

$\Omega \subset \mathbb{R}^2$: simply connected, bounded, smooth

$\widetilde{E}$: electric field, $\widetilde{H}$: magnetic field (3 components)

$\sigma(x_1, x_2)$: conductivity, $\mu(x_1, x_2)$: permeability

$\varepsilon(x_1, x_2)$: permittivity, $\gamma := \varepsilon + \frac{i\sigma}{\omega}$,

$\omega \neq 0, \in \mathbb{R}$: frequency

$$\begin{cases} \operatorname{rot} \widetilde{E} - i\omega\mu\widetilde{H} = 0, \\ \operatorname{rot} \widetilde{H} + i\omega\gamma\widetilde{E} = 0 \quad \text{in } \widetilde{\Omega}. \end{cases}$$

In waveguide $\Longrightarrow$

$$\widetilde{E}(x_1, x_2, x_3) = e^{hx_3} E(x_1, x_2),$$

$$\widetilde{H}(x_1, x_2, x_3) = e^{hx_3} H(x_1, x_2), \quad h \in \mathbb{C},$$

Example; $h = i\omega$.

Rewrite Maxwell's equations

$$L_{1,\mu,\gamma}(E,H) := \begin{pmatrix} \partial_2 E_3 - hE_2 \\ -\partial_1 E_3 + hE_1 \\ \partial_1 E_2 - \partial_2 E_1 \end{pmatrix} - i\omega\mu \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = 0, \quad \text{in } \Omega,$$

and

$$L_{2,\mu,\gamma}(x,D)(E,H) := \begin{pmatrix} \partial_2 H_3 - hH_2 \\ -\partial_1 H_3 + hH_1 \\ \partial_1 H_2 - \partial_2 H_1 \end{pmatrix} + i\omega\gamma \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = 0, \quad \text{in } \Omega.$$

$L := (L_{1,\mu,\gamma}, L_{2,\mu,\gamma})$,
$\nu = (\nu_1, \nu_2)$: outward unit normal to $\partial\Omega$,
$\vec{\nu} = (\nu_1, \nu_2, 0)$

$\widetilde{\Gamma} \subset \partial\Omega$: arbitrary subboundary

DN map:

$\Lambda_{\mu,\gamma} f = \vec{\nu} \times H$ on $\widetilde{\Gamma}$, where

$$\begin{cases} L_{\mu,\gamma}(E,H) = 0, \\ \vec{\nu} \times E = f, \quad \operatorname{supp} f \subset \widetilde{\Gamma}. \end{cases}$$

Technical remark.

- Solution (E, H) does not necessarily exist to the BVP!

  $D_{\mu,\gamma} := \{f \in H^1(\widetilde{\Gamma});\ \exists(E, H)\}$
- $\Lambda_{\mu_1,\gamma_1} = \Lambda_{\mu_2,\gamma_2}$ is defined if

  (i) $D_{\mu_1,\gamma_1} \subset D_{\mu_2,\gamma_2}$,

  (ii) $L_{\mu_1,\gamma_1}(E, H) = 0$, $\vec{\nu} \times E = f$, supp $f \subset \widetilde{\Gamma} \Longrightarrow$
  $\exists(\widehat{E}, \widehat{H})$ such that $L_{\mu_2,\gamma_2}(\widehat{E}, \widehat{H}) = 0$,
  $\vec{\nu} \times \widehat{E} = f$,
  $\vec{\nu} \times \widehat{H} = \vec{\nu} \times H$ on $\widetilde{\Gamma}$.

Theorem (uniqueness by local DN map)

Let $h^2 + \omega^2\gamma_j\mu_j \neq 0\ (\overline{\Omega})$, $j = 1, 2$,

$\mu_j > 0, \varepsilon_j > 0, \sigma_j \in C^5(\overline{\Omega})$.

Then

$$\begin{cases} \Lambda_{\mu_1,\gamma_1} = \Lambda_{\mu_2,\gamma_2}, & \\ \partial_\nu^k(\mu_1 - \mu_2) = \partial_\nu^k(\gamma_1 - \gamma_2) = 0, \quad k = 0, 1 \quad (\widetilde{\Gamma}) \end{cases}$$

$\Longrightarrow \mu_1 = \mu_2,\ \varepsilon_1 = \varepsilon_2,\ \sigma_1 = \sigma_2$ in Ω

- local DN map on arbitrary subboundary proves uniqueness for Maxwell's equations in waveguide

References on 3D Maxwell DN map with $\widetilde{\Gamma} = \partial\Omega$

- Caro-Ola-Salo 2009
- Ola-Päivärinta-Somersalo 1993

§2. 2D and 3D Calderón problems with local data

$$\begin{cases} (\Delta + q)u = 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = f \end{cases}$$

$\Gamma, \widetilde{\Gamma} \subset \partial\Omega$: subboundaries

$\Lambda_{\Gamma,\widetilde{\Gamma}} : \{f \in H^{\frac{1}{2}}(\partial\Omega) | \, \mathrm{supp} f \subset \Gamma\} \longrightarrow \frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}}$

- $\widetilde{\Gamma} = \Gamma = \partial\Omega$

Sylvester-Uhlmann: 1987 (2D),

Nachman: 1996 (3D)

References (incomplete) for $n = 2$

- For $\Gamma = \widetilde{\Gamma}$,

  $\Lambda_{\Gamma,\Gamma}$ uniquely determines

  - $q \in C^{2+\alpha}(\overline{\Omega})$

    (Imanuvilov-Uhlmann-Yamamoto:2010)

  - $q \in W_p^1(\Omega)$, $p > 2$

    $q \in C^{\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$ if $\Gamma = \widetilde{\Gamma} = \partial\Omega$

    (Imanuvilov-Yamamoto:2012)

- $q \in L^p(\Omega),\ p > 2$
  if $\Gamma = \widetilde{\Gamma} = \partial\Omega$ (Imanuvilov-Yamamoto:2013)

- $\Gamma \cap \widetilde{\Gamma} = \emptyset$: Imanuvilov-Uhlmann-Yamamoto (2011)
- partial Neumann-to-Dirichlet map on arbitrary subboundary: Imanuvilov-Uhlmann-Yamamoto (2015)

Other equations with $\widetilde{\Gamma} = \partial\Omega$

Navier-Stokes or Stokes equations

- 2D Case: Imanuvilov and Yamamoto (2015)
- 3D Case: Li-Wang (2007), Heck-Wang-Li (2007)

Global uniqueness for Lamé system

- 2D case: Imanuvilov and Yamamoto (2015)
- 3D case: unsolved

$n \geq 3$

- V. Isakov (2006): $\Gamma, \widetilde{\Gamma} \subset$ plane or sphere
- C. Kenig - J. Sjöstrand - G. Uhlmann (2007): $\Gamma \subset \partial\Omega$: arbitrary, $\widetilde{\Gamma} \supset\supset \partial\Omega \setminus \Gamma$

- Imanuvilov-Yamamoto (2013):
  3D cylindrical domain $\Omega = D \times (0, \ell)$:
  $\Gamma = \gamma \times (0, \ell)$ with arbitrary $\gamma \in \partial D$: global uniqueness by $\Lambda_{\partial\Omega\setminus\Gamma, \partial\Omega\setminus\Gamma}$
- Kenig-Salo (2013); on Riemannian manifold
- Imanuvilov-Yamamoto (2013): Uniqueness with some pairs of $\Gamma, \widetilde{\Gamma}$ with Int $(\Gamma) \cap$ Int $(\widetilde{\Gamma}) = \emptyset$
  Example 1: $\Gamma = \{x \in \partial\Omega | \vec{v} \cdot \nu \le 0\}$ and $\widetilde{\Gamma} = \{x \in \partial\Omega | \vec{v} \cdot \nu \ge 0\}$ with outward normal ν and fixed unit vector $\vec{v}$

Example 2: $\Gamma = \{x \in \partial\Omega | (x - x_0) \cdot \nu \leq 0\}$ and $\widetilde{\Gamma} = \{x \in \partial\Omega | (x - x_0) \cdot \nu \geq 0\}$ with outward normal ν and fixed point x_0

In 3D case, we need some geometric condition for

$\boldsymbol{\Gamma}, \widetilde{\boldsymbol{\Gamma}}$

Uniqueness for 3D Maxwell's equations in

$Q := \Omega \times (0, L)$

(Imanuvilov-Yamamoto 2014)

$\Gamma_0 \subset \partial\Omega$: arbitrary

$\Lambda_{\mu,\gamma} f = \vec{\nu} \times H$ on $\partial Q \setminus (\Gamma_0 \times (0, L))$ for

$$\begin{cases} L_{\mu,\gamma}(E, H) = 0 & \text{in } Q, \\ \vec{\nu} \times E = f, & \text{supp } f \subset \partial Q \setminus (\Gamma_0 \times (0, L)) \end{cases}$$

Theorem (Imanuvilov-Yamamoto 2014):

Let $\mu_j > 0, \varepsilon_j > 0, \sigma_j \in C^7(\overline{Q})$, $j = 1, 2$. Then

$$\begin{cases} \Lambda_{\mu_1,\gamma_1} = \Lambda_{\mu_2,\gamma_2}, \\ \partial_\nu^k(\mu_1 - \mu_2) = \partial_\nu^k(\gamma_1 - \gamma_2) = 0 \\ \qquad k = 0, 1 \quad \text{on } \partial Q \setminus (\Gamma_0 \times (0, L)) \end{cases}$$

$\Longrightarrow \mu_1 = \mu_2,\ \varepsilon_1 = \varepsilon_2,\ \sigma_1 = \sigma_2$

in (convex hull of Γ_0) $\times (0, L)$.

Our strategy

3D Maxwell in waveguide ~ 2D Maxwell

$\Longrightarrow$

We can expect uniqueness with arbitrary $\Gamma = \widetilde{\Gamma}$?

Similar to Imanuvilov-Yamamoto

§3. Proof for $h = 0$: simple case

- Perfect decoupling into two scalar elliptic equations
- DN map for Maxwell's equations $\Longrightarrow$
  DN map for the first scalar equation
  ND map for the second scalar equation

Maxwell's equations for $h = 0$

$$L_{1,\mu,\gamma}(E,H) := \begin{pmatrix} \partial_2 E_3 \\ -\partial_1 E_3 \\ \partial_1 E_2 - \partial_2 E_1 \end{pmatrix} - i\omega\mu \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = 0,$$

$$L_{2,\mu,\gamma}(E,H) := \begin{pmatrix} \partial_2 H_3 \\ -\partial_1 H_3 \\ \partial_1 H_2 - \partial_2 H_1 \end{pmatrix} + i\omega\gamma \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = 0, \quad \text{in } \Omega.$$

Here $\Omega \subset \mathbb{R}^2$, $E = (E_1, E_2)$, $H = (H_1, H_2)$,

$\gamma = \epsilon + \frac{i\sigma}{\omega}$

Dirichlet-to-Neumann map

$\Lambda_{\mu,\gamma} f = \vec{\nu} \times H$ on $\widetilde{\Gamma}$

where

$$\begin{cases} L_{\mu,\gamma}(x, D)(E, H) = 0 \quad \text{in } \Omega, \\ \vec{\nu} \times E|_{\partial\Omega \setminus \widetilde{\Gamma}} = 0, \\ \vec{\nu} \times E|_{\widetilde{\Gamma}} = f. \end{cases}$$

Theorem 0.

Let $\mu_j > 0, \varepsilon_j > 0, \sigma_j \in C^5(\overline{\Omega})$, $j = 1, 2$.

Then

$$\begin{cases} \Lambda_{\mu_1,\gamma_1} = \Lambda_{\mu_2,\gamma_2}, \\ \partial_\nu^k(\mu_1 - \mu_2) = \partial_\nu^k(\gamma_1 - \gamma_2) = 0, \quad k = 0, 1 \quad \text{on } \widetilde{\Gamma} \end{cases}$$

$\Longrightarrow \mu_1 = \mu_2$, $\varepsilon_1 = \varepsilon_2$, $\sigma_1 = \sigma_2$ in Ω.

Sketch of Proof

First Step: Decoupling.

$H_1 = \frac{1}{i\omega\mu}\partial_2 E_3,\ H_2 = -\frac{1}{i\omega\mu}\partial_1 E_3,$

$\partial_{x_1} H_2 - \partial_{x_2} H_1 = -i\omega\gamma E_3 \Longrightarrow$

$$P_{\mu,\gamma} E_3 := \mathrm{div}\left(\frac{1}{i\omega\mu}\nabla E_3\right) - i\omega\gamma E_3 = 0$$

Similarly

$$Q_{\mu,\gamma} H_3 := \mathrm{div}\left(\frac{1}{i\omega\gamma}\nabla H_3\right) - i\omega\mu H_3 = 0 \quad \text{in } \Omega$$

Second Step

$\Lambda_{\mu,\gamma}$ yields DN map and ND map:

- DN map: $\Lambda_{1,\mu,\gamma} f = \partial_\nu u|_{\widetilde{\Gamma}}$ where

$$\begin{cases} P_{\mu,\gamma} u = 0, \\ u|_{\widetilde{\Gamma}} = f, \\ u|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0 \end{cases}$$

- ND map: $\Lambda_{2,\mu,\gamma} f = v|_{\widetilde{\Gamma}}$ where

$$\begin{cases} Q_{\mu,\gamma} v = 0, \\ \partial_\nu v|_{\widetilde{\Gamma}} = f, \\ \partial_\nu v|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0. \end{cases}$$

- Imanuvilov-Uhlmann-Yamamoto (2010) $\Longrightarrow$ $\omega^2\gamma\mu + \frac{\Delta\sqrt{\mu}}{\sqrt{\mu}}$: uniquely determined by DN map
- Imanuvilov-Uhlmann-Yamamoto (Advances Math.: 2015) $\Longrightarrow$ $\omega^2\gamma\mu + \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$: uniquely determined by ND map

$$\begin{cases} \omega^2\gamma_1\mu_1 + \frac{\Delta\sqrt{\mu_1}}{\sqrt{\mu_1}} = \omega^2\gamma_2\mu_2 + \frac{\Delta\sqrt{\mu_2}}{\sqrt{\mu_2}}, \\ \omega^2\gamma_1\mu_1 + \frac{\Delta\sqrt{\gamma_1}}{\sqrt{\gamma_1}} = \omega^2\gamma_2\mu_2 + \frac{\Delta\sqrt{\gamma_2}}{\sqrt{\gamma_2}} \end{cases}$$

$U := (\sqrt{\mu_1} - \sqrt{\mu_2}, \sqrt{\gamma_1} - \sqrt{\gamma_2}) \Longrightarrow \exists A \in L^\infty(\Omega)^3$,
$\exists B \in L^\infty$ s.t.

$\Delta U + A \cdot \nabla U + BU = 0$ in Ω
and $U = \partial_\nu U = 0$ on $\widetilde{\Gamma}$.

Unique continuation yields $U = 0$ in Ω.

§4. Sketch of Proof for $h \neq 0$

Maxwell's equations in waveguide

$$\left\{\begin{array}{l} \begin{pmatrix} \partial_2 E_3 - hE_2 \\ -\partial_1 E_3 + hE_1 \\ \partial_1 E_2 - \partial_2 E_1 \end{pmatrix} - i\omega\mu \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = 0, \\ \begin{pmatrix} \partial_2 H_3 - hH_2 \\ -\partial_1 H_3 + hH_1 \\ \partial_1 H_2 - \partial_2 H_1 \end{pmatrix} + i\omega\gamma \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = 0, \quad \text{in } \Omega. \end{array}\right.$$

$\Longrightarrow$

$$\operatorname{div}\left(\left(\frac{1}{i\omega\mu}-\frac{h^2}{i\omega^3 g\gamma\mu^2}\right)\nabla E_3\right)$$

$$-\left(\nabla\left(\frac{h}{\omega^2\gamma\mu g}\right)\cdot\nabla^* H_3\right)-i\omega\gamma E_3=0, \quad \text{etc.}$$

Here $\nabla^* H_3 := (\partial_2 H_3, -\partial_1 H_3)$, $g := 1+\frac{h^2}{\omega^2\gamma\mu}$.

$\Longrightarrow$

$$\begin{cases} \Delta W + 2A\partial_z W + 2B\partial_{\overline{z}} W + QW = 0, \\ \mathcal{B}W|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0. \end{cases}$$

Here

$W := (E_3, H_3)$, $\mathcal{B}W := (E_3, \partial_\nu H_3)$, $\rho_1 := \dfrac{\omega\gamma}{i(h^2+\omega^2\gamma\mu)}$, $\rho_2 := \dfrac{\omega\mu}{i(h^2+\omega^2\gamma\mu)}$, $\rho_3 := \dfrac{h}{\omega^2\mu\gamma g}$,

$$A := \begin{pmatrix} \partial_{\bar z} \ln \rho_1 & -\frac{i}{\rho_1}\partial_{\bar z}\rho_3 \\ \frac{i}{\rho_2}\partial_{\bar z}\rho_3 & \partial_{\bar z} \ln \rho_2 \end{pmatrix},$$

$$B := \begin{pmatrix} \partial_z \ln \rho_1 & \frac{i}{\rho_1}\partial_z\rho_3 \\ -\frac{i}{\rho_2}\partial_z\rho_3 & \partial_z \ln \rho_2 \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{-i\omega\gamma}{\rho_1} & 0 \\ 0 & \frac{-i\omega\mu}{\rho_2} \end{pmatrix},$$

Maxwell DN map yields

DN map:

$$\widetilde{\Lambda}_{\mu,\gamma} f = (\partial_\nu E_3, H_3)$$

where

$$\begin{cases} \Delta W + 2A\partial_z W + 2B\partial_{\bar{z}} W + QW = 0, \\ \mathcal{B}W|_{\widetilde{\Omega}} = f, \\ \mathcal{B}W|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0. \end{cases}$$

Third Step:

Similarly to Imanuvilov and Yamamoto

(”Inverse Problems” 2012) for

weakly coupling ellitpic systems:

Construction of complex geometric optics solutions

by Carleman estimates $\Longrightarrow$

$\mu_1 = \mu_2$, $\varepsilon_1 = \varepsilon_2$, $\sigma_1 = \sigma_2$ in Ω

Supplementary newest reference

Imanuvilov and Yamamoto (12 May 2016)

$u := (u_1, ..., u_N)$, A, B, Q: $N \times N$ matrices

$$\Delta u + 2A\partial_z u + 2B\partial_{\bar{z}} u + Qu = 0 \quad \text{in } \Omega \subset \mathbb{R}^2$$

DN map: $\Lambda_{A,B,Q} f = \partial_\nu u|_{\widetilde{\Gamma}}$ where

$$\begin{cases} \Delta u + 2A\partial_z u + 2B\partial_{\bar{z}} u + Qu = 0, \\ u|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0, \quad u|_{\widetilde{\Gamma}} = f. \end{cases}$$

Theorem (gauge uniqueness)

Let $A_j, B_j \in C^{5+\alpha}(\overline{\Omega})$, $Q_j \in C^{4+\alpha}(\overline{\Omega})$, $j = 1, 2$, the elliptic operators and the adjoints have not the zero eigenvalue.

Then $\Lambda_{A_1,B_1,Q_1} = \Lambda_{A_2,B_2,Q_2}$ if and only if

$$A_1 = A_2, \quad B_1 = B_2 \quad \text{on } \widetilde{\Gamma},$$

and $\exists$ invertible matrix $\mathbf{R} \in C^{6+\alpha}(\overline{\Omega})$ such that

$$\mathbf{R}|_{\widetilde{\Gamma}} = I, \quad \partial_{\vec{\nu}}\mathbf{R}|_{\widetilde{\Gamma}} = 0,$$

$$A_2 = 2\mathbf{R}^{-1}\partial_{\bar{z}}\mathbf{R} + \mathbf{R}^{-1}A_1\mathbf{R}, \quad B_2 = 2\mathbf{R}^{-1}\partial_z\mathbf{R} + \mathbf{R}^{-1}B_1\mathbf{R}$$

$$Q_2 = \mathbf{R}^{-1}Q_1\mathbf{R} + \mathbf{R}^{-1}\Delta\mathbf{R} + 2\mathbf{R}^{-1}A_1\partial_z\mathbf{R} + 2\mathbf{R}^{-1}B_1\partial_{\bar{z}}\mathbf{R} \quad \text{in } \Omega$$

Ref: Eskin 2001: dimensions ≥ 3, $\widetilde{\Gamma} = \partial\Omega$

Thank you very much!