Thresholds in the spectrum of Hamiltonians with translation invariant magnetic fields

Nicolas Popoff
Institut Mathématique de Bordeaux

New trends in theoretical and numerical analysis of waveguides
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Fibered operators

Context:
- Models with invariance properties are reduced to lower dimensional problems.
  - Periodic invariance $\implies$ Floquet-Bloch transform (crystalline structures).
  - Translation invariance $\implies$ partial Fourier transform (waveguide, stratified media, magnetic field).
- In general, $H$ having invariance properties and $U$ a unitary transform:
  \[ UHU^* = \int \oplus \eta(k) dk. \]
- Fiber operators $\eta(k)$ in transversal directions (most of the time) have confining properties, and therefore compact resolvent.

The band functions:
- Their spectra $k \mapsto (\lambda_n(k))_{n \geq 1}$ are the bands functions. The spectrum of $H$ is
  \[ \sigma(H) = \bigcup_{n \geq 1} \text{Ran} \lambda_n. \]
- Understanding of the $\lambda_n$ is crucial if you want to understand (for example):
  - Quantum transport properties of the system in the invariance direction.
  - How the system react under perturbations.
A (very) explicit case

- Laplacian in straight cylindrical waveguide
  \[ \Omega = \omega \times \mathbb{R} + b.c. \]
- \((\mu_n)_{n \geq 1}\) the eigenvalues of the Laplacian in the section \(-\Delta^\omega\). Explicit fibers
  \[ \eta(k) = -\Delta^\omega + k^2 \quad \text{and} \quad \lambda_n(k) = \mu_n + k^2. \]
- \(\mu_n\) are thresholds (or cut-off frequencies).

and its perturbations:
- Electric potential and/or Magnetic field.
- Bending and twisting.
- With obstacles or windows: more difficult framework.

Think also of stratified media!!
The magnetic Laplacian

Schrödinger operator with magnetic field:

\[ H = (-i\nabla - A)^2, \quad B = \text{curl} \, A \quad \text{the magnetic field.} \]

A model case:

**Landau Hamiltonian:** \( \Omega = \mathbb{R}^2 \) and \( B = 1 \).

The spectrum are the **Landau Levels** \( E_n := (2n - 1)_{n \geq 1} \), with infinite multiplicity. The eigenspaces are explicit using the **Hermite’s functions**.

2D Iwatsuka and hard walls models:

- \( \Omega = I \times \mathbb{R}, \ I \subset \mathbb{R} \) and \( B = B(x) \): invariance along \( y \).
  
  \[ H = -\partial_x^2 + (-i\partial_y - a(x))^2, \quad a'(x) = B(x) \quad \text{and b.c.} \]

- **Partial Fourier transform** along \( y \): \( U \varphi(x, k) = \int_{\mathbb{R}} e^{-iky} \varphi(x, y)dy \).

  Fibers: \( \mathfrak{h}(k) = -\partial_x^2 + (k - a(x))^2 \) acting on \( L^2(I) + \text{b.c.} \).
State of the art and basis for these models

Two dimensional magnetic models

- Iwatsuka model: \( I = \mathbb{R} \) and (roughly) \( B \) regular positive non constant and having limits as \(|x| \to +\infty\). The spectrum of \( H \) is absolutely continuous [Iwatsuka 85]. The propagation occurs on the form on edge states [Mantoiu-Purice 97] (to be continued).
- One or two hard walls, \( B = 1 \) and \( I = \mathbb{R}_+ \) or \((-L, L)\) [de Biève-Pulé], [Hislop-Soccorsi], [Geiler-Sanatorov], [Raikov-Briet-Soccorsi], [Bruneau-Miranda-Raikov]...).

Properties of the fibers:

- \( h(k) \) forms a Kato family with compact resolvent, the band functions \( k \mapsto \lambda_n(k) \) are simple and analytic. If they are proper \((\lim_{|k| \to \infty} \lambda_n(k) = +\infty)\), these models enters the theory of [Gérard-Nier 98].
- Denote by \( u_n(x, k) \) associated normalized eigenfunction. Fourier decomposition and projector on the \( n \)-th harmonics: given \( \varphi \in L^2(\mathbb{R}^2_+) \),

\[
\varphi_n(k) = \int_I \hat{\varphi}_y(x, k) u_n(x, k) \, dx \quad \text{and} \quad \pi_n(\varphi)(x, y) = \int_k \varphi_n(k) e^{iky} u_n(x, k) \, dk
\]
Velocity operator and thresholds

The position observable is the multiplication by $y$:

- Its time evolution is $y(t) = e^{-itH}ye^{itH}$. Its time derivative is the current observable:
  
  $$y'(t) = [H, iy(t)] \quad \text{with} \quad [H, iy] = -i\partial_y - a(x) = J_y.$$

- Well known Feynman-Hellmann formula:
  
  $$\langle J_y \pi_n(\varphi), \pi_n(\varphi) \rangle_{L^2} = \int_k \lambda'_n(k) |\varphi_n(k)|^2 dk$$

- The velocity operator acts as the multiplication by the $(\lambda'_n(k))_{n \geq 1}$ in suitable Fourier basis.

The thresholds:

- Heuristic definition: spectral values $E$ such that
  
  $$E \in T \iff \exists (n, k) \in \mathbb{N}^* \times \mathbb{R}, \quad |\lambda'_n(k)| << 1 \text{ and } \lambda_n(k) \sim E.$$

- Typically: critical point of $\lambda_n$, but also...
Example of band functions with non-reached thresholds

Figure: Example of band functions. The thresholds are the Landau levels.
Outside the thresholds

Mourre estimates:

- Let $I \subset \mathbb{R}$ be an interval without threshold. Mourre estimate does exist:

$$\exists c(I) > 0, \forall \chi \in C_0^\infty(I), \quad \chi(H)[H, A_I]\chi(H) \geq c(I)\chi(H)^2.$$ 

- Key ingredient: lower bound for $\lambda'_n(k)$ when $k \in \lambda_n^{-1}(I)$. Used many times, particular case from [Gérard-Nier 97].

Consequence:

- Limit absorption principle outside the thresholds.
- Description of scattering theory, resonances and eigenvalues created by suitable perturbations.
Edge states and problematics

The edge states:

- Given a quantum states $\varphi = \sum_n \pi_n \varphi$. The group velocity of its harmonics $\pi_n \varphi$ are the values $\lambda'_n(k)$ when $k \in \text{supp}(\varphi_n)$.
- Let $I \subset \sigma(H)$ without threshold and assume $\lambda'_n$ has constant sign, so that:

$$\exists c(I) > 0, \; \forall \varphi \in \text{Ran} \; P_I, \; \forall k \in \text{supp}(\varphi_n(k)), \quad |\lambda'_n(k)| \geq c(I)$$

where $P_I$ is the spectral projector on $I$.
- Such $\varphi$ are called edge states: in general they are localized in $x$.
- Direction of propagation along $y$ is linked to the sign of $\lambda'_n(k)$.
- Easily adapted to non-monotonous band functions.

What happens when $I$ has thresholds?

- Standard Mourre estimates does not hold anymore.
- We will focus on non reached thresholds $E$:

$$\lim_{k \to \infty} \lambda_n(k) = E.$$ 

Physically this corresponds to the presence of bulk states.
Problematics

Some questions:

- Can you describe the limit of the band functions, depending on the model?
- Can you describe the associated bulk states?
- Can you give the number of eigenvalues for suitable perturbation?
- Can you do an absorption principle? Can you describe the behavior of the resolvent near thresholds?
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A well known model: the Landau Hamiltonian in a half-plane

Operator arising in quantum Hall effect modelling and surface superconductivity:

$$\Omega = \mathbb{R}_+ \times \mathbb{R}$$ with constant magnetic field and Dirichlet OR Neumann b.c.

Fiber: $$\eta(k) = -\partial_x^2 + (x - k)^2$$ on $$L^2(\mathbb{R}_+)$$ with D/N b.c. at $$x = 0$$.

Proposition [Plenty of people]

1. For Dirichlet: The band functions $$k \mapsto \lambda^D_n(k)$$ are decreasing on $$\mathbb{R}$$.

2. For Neumann: The band functions $$k \mapsto \lambda^N_n(k)$$ admit a unique non degenerate minimum and

$$(\lambda^N_n)'(k) = (\lambda^N_n(k) - k^2)u'_n(0, k)^2.$$

3. In both cases: $$\lim_{k \to -\infty} \lambda_n^{D/N}(k) = +\infty$$ and

$$\lambda_n^{D/N}(k) \xrightarrow{k \to +\infty} E_n \pm C_n k^{2n-1} e^{-k^2} + O(k^{2n-3} e^{-k^2}).$$
Edge states and Bulk states (Dirichlet case)

Bulk states:
- Focus on $I_n(\delta) = (E_n, E_n + \delta)$, $0 < \delta \ll 1$.
- Can you describe the element of $\text{Ran } P_{I_n(\delta)} \cap \text{Ran } \pi_n := X^\delta_n$?

Figure: The energy interval $E_1(\delta)$. 
Theorem [Hislop-Soccorsi-Popoff 15]

All $\varphi \in X^n_\delta$ has small current:

$$|\langle J_y \varphi, \varphi \rangle| \leq (2\delta \sqrt{|\log \delta|} + O\left(\frac{\log|\log \delta|}{\sqrt{|\log \delta|}}\right))\|\varphi\|_{L^2}^2,$$

and is localized far from the edge:

$$\forall \eta \in (0, 1), \int_{x=0}^{(1-\eta)\sqrt{|\log \delta|}} \|\varphi(x, \cdot)\|_{L^2(\mathbb{R})}^2 \, dx \leq c_n(\eta)\delta^\eta \log \delta \frac{2n-1}{2}(1-\eta^2)\|\varphi\|_{L^2}^2.$$

- The estimate on the current just need precise estimate of $\lambda'_n(k)$ as $k \to +\infty$.
- Localization requires more careful analysis in phase space and informations on $u_n(x, k)$.
- The method easily adapts.
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The case of reached thresholds

General problematics:
Let $V \geq 0$, with $\lim V(x, y) = 0$ be a relatively compact perturbation of $H$. The infimum of the essential spectrum is still $\inf \lambda_1$. But isolated eigenvalues (trapped modes) can appear!!

$$\mathcal{N}(r) := \#\{\sigma(H - V) \cap (-\infty, \inf \lambda_1 - r)\}$$

Objectif: behavior of $\mathcal{N}(r)$ as $r \to 0$.

**Theorem [Raikov 90’]**
Assume that $k \mapsto \lambda_1(k)$ admits a unique non-degenerate minimum in $k_*$. Then $\mathcal{N}(r)$ behaves like the number of negative eigenvalues of

$$-\mu \partial_y^2 - v_{\text{eff}}$$

with $\mu := \frac{1}{2} \lambda_1''(k_*) > 0$ and $v_{\text{eff}}(y) = \int_x V(x, y) u_1(x, k_*)^2 dx$.

Applied to $H^N - V$ in [Bruneau-Miranda-Raikov 14] and [Hislop-Soccorsi 14].
Results for the half-plane Landau Hamiltonian

In the half-plane model, no existence of an “effective mass”! But still:

**Theorem [Bruneau-Miranda-Raikov 13]**

In the case $H = H^D$, $\mathcal{N}(r)$ behaves like the number of negative eigenvalues of

$$\lambda_1^D - E_1 - \mathcal{V}$$

with $\mathcal{V}$ the anti-Wick symbol of $V$:

$$\mathcal{V} := \int_{x,\xi} V(x, \xi) \mathcal{P}_{x,\xi} dx d\xi$$

with

$$\mathcal{P}_{x,\xi} := |\psi_{x,\xi} \rangle <\psi_{x,\xi}|, \quad \psi_{x,\xi}(k) := e^{ik\xi} e^{-\frac{(x-k)^2}{2}}$$

**Corollary:** for non-zero compactly supported potential $V$, $H^D - V$ has infinite discrete spectrum since $H^N - V$ has finite discrete spectrum.
3d model

An analogous 3d model: Schrödinger operator in $\mathbb{R}^3$ with magnetic field created by an infinite rectilinear wire bearing a constant current:

$$B(x, y, z) = \frac{1}{r^2}(-y, x, 0), \quad r := \sqrt{x^2 + y^2}$$

$r$ : distance to the wire.

$$H_A := (-i\nabla - A)^2 = D_x^2 + D_y^2 + (D_z - \log r)^2,$$

Fibration:

- Cylindrical coordinates + angular FT $\Phi$ + $z$-FT $\mathcal{F}_3$:

$$\Phi \mathcal{F}_3 H_A \mathcal{F}_3^* \Phi^* := \sum_{m \in \mathbb{Z}} \int_{k \in \mathbb{R}} g_m(k) dk$$

$$g_m(k) := -\partial_r^2 + \frac{m^2 - \frac{1}{4}}{r^2} + (\log r - k)^2, \quad \text{on } L^2([0, +\infty), dr)$$

- $\sigma(g_m(k)) = \{\lambda_{n,m}(k), \ n \in \mathbb{N}^*, m \in \mathbb{Z}, k \in \mathbb{R}\}.$

- $\lambda'_{m,n}(k) < 0$ and $\lambda_{m,n}(k) \to 0$ as $k \to +\infty$. [Yafaev 03, Yafaev 08].
Figure: Band functions for a magnetic field created by an infinite wire.
Theorem (B.-Popoff ’14)

For all \((m, n) \in \mathbb{Z} \times \mathbb{N}^*\), there exist constants \(C_{m,n} > 0\) and \(k_0 \in \mathbb{R}\) such that

\[
\forall k \geq k_0, \quad |\lambda_{m,n}(k) - (2n - 1)e^{-k} + (m^2 - \frac{1}{4} - \frac{n(n-1)}{2})e^{-2k}| \leq C_{m,n}e^{-5k/2}
\]
Results for suitable perturbations

Theorem (Bruneau-P. ’14)
Assume $V(r, z) \geq \langle r \rangle^{-\alpha} v_\perp(z)$, $\alpha > 0$ with
(i) $\alpha < \frac{1}{2}$ and $v_\perp \in L^1(\mathbb{R})$ such that $\int_{\mathbb{R}} v_\perp(z)dz > 0$.

or
(ii) $v_\perp \geq C\langle z \rangle^{-\gamma}$ with $\gamma > 0$ and $\alpha + \frac{\gamma}{2} < 1$

Then $H_A - V$ have a infinite number of negative eigenvalues.

Theorem (Bruneau-P. ’14)
Assume $V$ is a relatively compact perturbation of $H_A$ such that
$V(x, y, z) \leq \langle (x, y) \rangle^{-\alpha} v_\perp(z)$, with $\alpha > 1$ and $0 \leq v_\perp \in L^1(\mathbb{R})$.

Then $H_A - V$ have, at most, a finite number of negative eigenvalues.

Note that in some case, $H_A - V$ have finite discrete spectrum since $-\Delta - V$ has not!
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Limiting absorption principle (LAP)

LAP for fibered operator

- **Outside the thresholds:** Well known LAP on weighted spaces

\[ L^{2,s} := \{ \varphi \in L^2, (1 + y^2)^{s/2} \varphi \in L^2 \}. \]

- On non-degenerate critical point \( \lambda'_n(k_*) = 0 \), LAP on the \( NH^s \) spaces [Croc-Dermenjian 95]

\[ NH^s := \{ \varphi \in L^{2,s}, \varphi_n(k_*) = 0 \}. \]

- Extended to any critical point ("reached thresholds") in [Soccorsi 99].

New framework for non-reached thresholds:

- LAP for the Landau Dirichlet Hamiltonian in a half-plane [P-Soccorsi 15] at Landau level.

- The absorption spaces require exponential decay on \( k \mapsto \varphi_n(k) \).

- Associate functions are localized near the edge though phase-space analysis.

This general method is easily adapted to any non-reached threshold!
To go further

Expansion of the resolvent

- Expansion of the resolvent at reached threshold (critical point of the band functions) is done in general in [Gérard 90].
- Description of the resonances for small electrical perturbations follows [Grigis-Klopp 95]. You need to know some algebraic geometry to extend singular integral on the universal covering of the resolvent set.
- Non-reached thresholds needs a new framework and will give rise to new singularity of the resolvent.

Example of the Landau Hamiltonian in a half-plane:
\[
\lambda_n(k) - E_n \sim k^{2n-1} e^{-k^2} \quad \Rightarrow \quad \text{essential singularity of the symbol at } \infty.
\]

- Global expansion: complexify the frequency \( k \).

And perturbations

- Once you have extended the resolvent around the thresholds, study eigenvalues and resonances for perturbations (electric potential, deformation of the boundary, obstacles...).
- Numerics will help!!
Iwatsuka’s magnetic steps

Piecewise constant magnetic field

- Snakes’ orbit in 2D electron gas [Peeter-Reijniers 2000].
- Magnetic field in $\mathbb{R}^2$:

$$B(x, y) = B(x) = \begin{cases} b_1 & \text{if } x < 0 \\ b_2 & \text{if } x > 0 \end{cases}$$

- When $0 < b_1 < b_2$, the band function are increasing [Hislop-Soccorsi 13].
- $b_1 = 0 \implies h(k)$ does not have compact resolvent: special treatment.
- By scaling, we are reduced to $b_1 < 0 < b_2 = 1$.

Limit for large frequencies:

- $\lambda_n(k) \to +\infty$ as $k \to -\infty$. Let $\mathcal{L}$ be the set of the Landau levels:

$$\lambda_n(k) \to +\infty \text{ as } k \to -\infty.$$  

As $k \to +\infty$, the limits of the band functions is precisely $\mathcal{T} := \mathcal{L} \cup |b_1|\mathcal{L}$.

- The dispersion curves depend on $r := \frac{b_1}{b_2}$.
- Disjonction depending on wether $r$ has the form $-\frac{2n-1}{2m-1}$ or not.
Courbes de dispersions du modèle d’Iwatsuka

\[ b_1 = -1, \quad b_2 = 1, \quad r = -\frac{1}{1} \]

En abscisse : \( k \) paramètre de Fourier dual de \( y \).
Courbes de dispersions du modèle d’Iwatsuka

\[ b_1 = -1.1, \quad b_2 = 1, \quad r = -\frac{11}{10} \]

En abscisse : \( k \) paramètre de Fourier dual de \( y \).
Courbes de dispersions du modèle d’Iwatsuka

\[ b_1 = -1.2, \quad b_2 = 1, \quad r = -\frac{6}{5} \]

En abscisse : \( k \) paramètre de Fourier dual de \( y \).
Courbes de dispersions du modèle d’Iwatsuka

\[ b_1 = -1.3, \quad b_2 = 1, \quad r = -\frac{13}{10} \]

En abscisse : \( k \) paramètre de Fourier dual de \( y \).
Courbes de dispersions du modèle d’Iwatsuka

\[ b_1 = -1.4, \quad b_2 = 1, \quad r = -\frac{7}{5} \]

En abscisse : \( k \) paramètre de Fourier dual de \( y \).
Courbes de dispersions du modèle d’Iwatsuka

\[ b_1 = -1.5, \quad b_2 = 1, \quad r = -\frac{3}{2} \]

En abscisse : \( k \) paramètre de Fourier dual de \( y \).
Courbes de dispersions du modèle d’Iwatsuka

\[ b_1 = -1.6, \quad b_2 = 1, \quad r = -\frac{8}{5} \]

En abscisse : \( k \) paramètre de Fourier dual de \( y \).
Courbes de dispersions du modèle d’Iwatsuka

\[ b_1 = -1.667, \quad b_2 = 1, \quad r = -\frac{5}{3} \]

En abscisse : \( k \) paramètre de Fourier dual de \( y \).
Asymptotics of the band function

**Theorem [Hislop-Persson-Popoff-Raymond 16]**

1. **(Non splitting case).** Let $E \in \mathcal{T} \setminus (\mathcal{L} \cap |b_1|\mathcal{L})$. Then there exists a unique band function converging to $E$, and the convergence is by above:

$$\lambda_{p_n}(k) = \begin{cases} E - C_n k^{2n-3} e^{-k^2} (1 + o(1)) & \text{if } E = 2n - 1 \\ E - \widetilde{C}_n k^{2n-3} e^{-k^2/|b_1|} (1 + o(1)) & \text{if } E = |b_1|(2n - 1) \end{cases}$$

2. **(Splitting case).** Let $E \in \mathcal{L} \cap |b_1|\mathcal{L}$. Write $E = E_n = |b_1|E_m$. Two band functions converge toward $E$, one from below and one from above. The threshold is asymptotically degenerate and

$$\begin{cases} \lambda_{p_n}(k) = E - C_n k^{2n-3} e^{-k^2} (1 + o(1)) \\ \lambda_{p_n+1}(k) = E + \widetilde{C}_m k^{2m+1} e^{-k^2/b} (1 + o(1)) \end{cases}$$
Consequences:
- When $|b_1|$ has not the form $-\frac{2n-1}{2m-1}$, all the band functions have a global minimum. The quantum transport can occur in two opposite directions since $\lambda_n$ are not monotonous.
- Precise asymptotics is needed for precise description of the threshold.

Method
- The large $k$ limit is equivalent to a 1d semi-classical problem with double wells:

$$\mathfrak{h}(k) \equiv k^2 \left( k^{-4} D_X^2 + V(X) \right), \quad V(X) = \begin{cases} (xb_1 - 1)^2, & x < 0 \\ (x - 1), & x > 0 \end{cases}$$

- Better to use special functions: $\lambda$ is an eigenvalue of $\mathfrak{h}(k)$ iff

$$U(-\frac{\lambda}{2b}, -\sqrt{2} \frac{k}{\sqrt{b}}) U'(-\frac{\lambda}{2}, -\sqrt{2}k) + \sqrt{b} U'(-\frac{\lambda}{2b}, -\sqrt{2} \frac{k}{\sqrt{b}}) U(-\frac{\lambda}{2}, -\sqrt{2}k) = 0$$

where $U$ is the first Weber parabolic cylindrical function.