Thresholds in the spectrum of Hamiltonians with translation invariant magnetic fields

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Plan

Fibered operators

Context:

- Models with invariance properties are reduced to lower dimensional problems.
  Periodic invariance $\Longrightarrow$ Floquet-Bloch transform (cristallin structures).
  Translation invariance $\Longrightarrow$ partial Fourier transform (waveguide, stratified media, magnetic field).
- In general, H having invariance properties and U a unitary transform:

$$UHU^* = \int^{\oplus} \mathfrak{h}(k) \mathrm{d}k.$$

- Fiber operators $\mathfrak{h}(k)$ in transversal directions (most of the time) have confining properties, and therefore compact resolvent.

The band functions:

- Their spectra $k \mapsto (\lambda_n(k))_{n\geq 1}$ are the bands functions. The spectrum of H is

$$\sigma(H) = \overline{\bigcup_{n\geq 1} \mathrm{Ran}\, \lambda_n}.$$

- Understanding of the λ_n is crucial if you want to understand (for example):
  Quantum transport properties of the system in the invariance direction.
  How the system react under perturbations.

A (very) explicit case

- Laplacian in straight cylindrical waveguide

$$\Omega = \omega \times \mathbb{R} \quad + b.c.$$

- $(\mu_n)_{n\geq 1}$ the eigenvalues of the Laplacian in the section $-\Delta^{\omega}$. Explicit fibers

$$\mathfrak{h}(k) = -\Delta^{\omega} + k^2 \quad \text{and} \quad \lambda_n(k) = \mu_n + k^2.$$

- μ_n are thresholds (or cut-off frequencies).

and its perturbations:[Plenty of people]

- Electric potential and/or Magnetic field.
- Bending and twisting.
- With obstacles or windows : more difficult framework.

Think also of stratified media!!

The magnetic Laplacian

Schrödinger operator with magnetic field:

$$H = (-i\nabla - A)^2, \quad B = \operatorname{curl} A \quad \text{the magnetic field.}$$

A model case:

Landau Hamiltonian: $\Omega = \mathbb{R}^2$ and $B = 1$.

The spectrum are the Landau Levels $E_n := (2n-1)_{n\geq 1}$, with infinite multiplicity.
The eigenspaces are explicit using the Hermite's functions.

2D Iwatsuka and hard walls models:

- $\Omega = I \times \mathbb{R}$, $I \subset \mathbb{R}$ and $B = B(x)$: invariance along y.

$$H = -\partial_x^2 + (-i\partial_y - a(x))^2, \quad a'(x) = B(x) \text{ and } b.c.$$

- Partial Fourier transform along y: $U\varphi(x,k) = \int_{\mathbb{R}} e^{-iky}\varphi(x,y)\mathrm{d}y$.

Fibers: $\mathfrak{h}(k) = -\partial_x^2 + (k - a(x))^2$ acting on $L^2(I) +$ b.c.

State of the art and basis for these models

Two dimensional magnetic models

- Iwatsuka model: $I = \mathbb{R}$ and (roughly) B regular positive non constant and having limits as $|x| \to +\infty$. The spectrum of H is absolutely continuous [Iwatsuka 85]. The propagation occurs on the form on edge states [Mantoiu-Purice 97] (to be continued).
- One or two hard walls, $B = 1$ and $I = \mathbb{R}_+$ or $(-L, L)$ [de Bièvre-Pulé], [Hislop-Soccorsi], [Geiler-Sanatorov], [Raikov-Briet-Soccorsi], [Bruneau-Miranda-Raikov]...).

Properties of the fibers:

- $\mathfrak{h}(k)$ forms a Kato family with compact resolvent, the band functions $k \mapsto \lambda_n(k)$ are simple and analytic. If they are proper ($\lim_{|k|\to\infty} \lambda_n(k) = +\infty$), these models enters the theory of [Gérard-Nier 98].
- Denote by $u_n(x,k)$ associated normalized eigenfunction. Fourier decomposition and projector on the n-th harmonics: given $\varphi \in L^2(\mathbb{R}^2_+)$,

$$\varphi_n(k) = \int_I \widehat{\varphi}_y(x,k) u_n(x,k) \mathrm{d}x \quad \text{and} \quad \pi_n(\varphi)(x,y) = \int_k \varphi_n(k) e^{iky} u_n(x,k) \mathrm{d}k$$

Velocity operator and thresholds

The position observable is the multiplication by y:

- Its time evolution is $y(t) = e^{-itH} y e^{itH}$. Its time derivative is the current observable:

$$y'(t) = [H, iy(t)] \quad \text{with } [H, iy] = -i\partial_y - a(x) = J_y.$$

- Well known Feynman-Hellmann formula:

$$\langle J_y \pi_n(\varphi), \pi_n(\varphi)\rangle_{L^2} = \int_k \lambda_n'(k) |\varphi_n(k)|^2 dk$$

- The velocity operator acts as the multiplication by the $(\lambda_n'(k))_{n\geq 1}$ in suitable Fourier basis.

The thresholds:

- Heuristic definition : spectral values E such that

$$E \in \mathcal{T} \iff \exists (n,k) \in \mathbb{N}^* \times \mathbb{R}, \quad |\lambda_n'(k)| << 1 \text{ and } \lambda_n(k) \sim E.$$

- Typically: critical point of λ_n, but also...

Example of band functions with non-reached thresholds

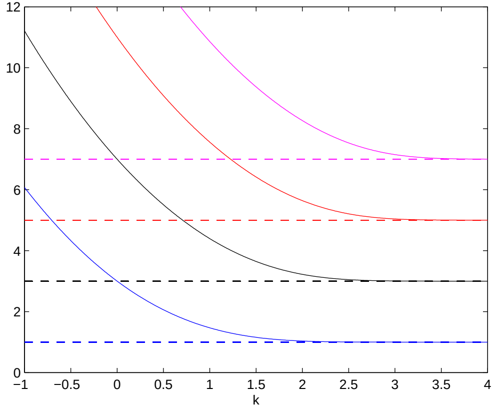


Figure: Example of band functions. The thresholds are the Landau levels.

Outside the thresholds

Mourre estimates:

- Let $I \subset \mathbb{R}$ be an interval without threshold. Mourre estimate does exist:

$$\exists c(I) > 0, \forall \chi \in C_0^\infty(I), \quad \chi(H)[H, A_I]\chi(H) \geq c(I)\chi(H)^2.$$

- Key ingredient: lower bound for $\lambda_n'(k)$ when $k \in \lambda_n^{-1}(I)$. Used many times, particular case from [Gérard-Nier 97].

Consequence:

- Limit absorption principle outside the thresholds.
- Description of scattering theory, resonances and eigenvalues created by suitable perturbations.

Edge states and problematics

The edge states:

- Given a quantum states $\varphi = \sum_n \pi_n \varphi$. The group velocity of its harmonics $\pi_n \varphi$ are the values $\lambda_n'(k)$ when $k \in \operatorname{supp}(\varphi_n)$.
- Let $I \subset \sigma(H)$ without threshold and assume λ_n' has constant sign, so that:

$$\exists c(I) > 0, \ \forall \varphi \in \operatorname{Ran} P_I, \ \forall k \in \operatorname{supp}(\varphi_n(k)), \quad |\lambda_n'(k)| \geq c(I)$$

where P_I is the spectral projector on I.

- Such φ are called edge states: in general they are localized in x.
- Direction of propagation along y is linked to the sign of $\lambda_n'(k)$.
- Easily adapted to non-monotonous band functions.

What happens when I has thresholds?

- Standard Mourre estimates does not hold anymore.
- We will focus on non reached thresholds E:

$$\lim_{k \to \infty} \lambda_n(k) = E.$$

Physically this corresponds to the presence of *bulk states*.

Problematics

Some questions:

- Can you describe the limit of the band functions, depending on the model?
- Can you describe the associated bulk states?
- Can you give the number of eigenvalues for suitable perturbation?
- Can you do an absorption principle? Can you describe the behavior of the resolvent near thresholds?

Plan

A well known model: the Landau Hamiltonian in a half-plane

Operator arising in quantum Hall effect modelling and surface superconductivity:

$\Omega = \mathbb{R}_+ \times \mathbb{R}$ with constant magnetic field and Dirichlet OR Neumann b.c.

Fiber: $\mathfrak{h}(k) = -\partial_x^2 + (x-k)^2$ on $L^2(\mathbb{R}_+)$ with D/N b.c. at $x = 0$.

Proposition [Plenty of people]

1. For Dirichlet: The band functions $k \mapsto \lambda_n^D(k)$ are decreasing on $\mathbb{R}$.
2. For Neumann: The band functions $k \mapsto \lambda_n^N(k)$ admit a unique non degenerate minimum and

$$(\lambda_n^N)'(k) = (\lambda_n^N(k) - k^2)u_n'(0,k)^2.$$

3. In both cases: $\lim_{k\to-\infty} \lambda_n^{D/N}(k) = +\infty$ and

$$\lambda_n^{D/N}(k) \underset{k\to+\infty}{=} E_n \pm C_n k^{2n-1} e^{-k^2} + O(k^{2n-3}e^{-k^2}).$$

Edge states and Bulk states (Dirichlet case)

Bulk states:

- Focus on $I_n(\delta) = (E_n, E_n + \delta)$, $0 < \delta << 1$.
- Can you describe the element of $\mathrm{Ran}\, P_{I_n(\delta)} \cap \mathrm{Ran}\, \pi_n := X_n^\delta$?

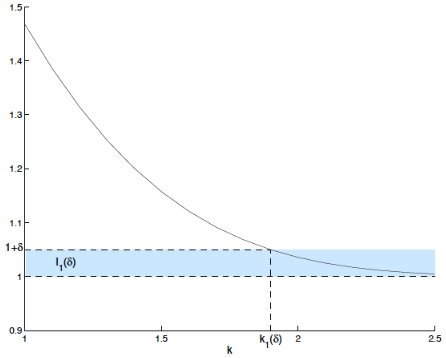


Figure: The energy interval $E_1(\delta)$.

Result

Theorem [Hislop-Soccorsi-Popoff 15]

All $\varphi \in X_n^\delta$ has small current:

$$|\langle J_y \varphi, \varphi \rangle| \leq (2\delta\sqrt{|\log \delta|} + O(\frac{\log|\log \delta|}{\sqrt{|\log \delta|}}))\|\varphi\|_{L^2}^2,$$

and is localized far from the edge:

$$\forall \eta \in (0,1), \quad \int_{x=0}^{(1-\eta)\sqrt{|\log \delta|}} \|\varphi(x,\cdot)\|_{L^2(\mathbb{R})}^2 \mathrm{d}x \leq c_n(\eta)\delta^{\eta^2}|\log \delta|^{\frac{2n-1}{2}(1-\eta^2)}\|\varphi\|_{L^2}^2.$$

- The estimate on the current just need precise estimate of $\lambda_n'(k)$ as $k \to +\infty$.
- Localization requires more careful analysis in phase space and informations on $u_n(x,k)$.
- The method easily adapts.

Plan

The case of reached thresholds

General problematics:
Let $V \geq 0$, with $\lim V(x,y) = 0$ be a relatively compact perturbation of H. The infimum of the essential spectrum is still $\inf \lambda_1$. But isolated eigenvalues (trapped modes) can appear!!

$$\mathcal{N}(r) := \#\{\sigma(H - V) \cap (-\infty, \inf \lambda_1 - r)\}$$

Objectif: behavior of $\mathcal{N}(r)$ as $r \to 0$.

Theorem [Raikov 90']

Assume that $k \mapsto \lambda_1(k)$ admits a unique non-degenerate minimum in k_*. Then $\mathcal{N}(r)$ behaves like the number of negative eigenvalues of

$$-\mu\partial_y^2 - v_{eff}$$

with $\mu := \frac{1}{2}\lambda_1''(k_*) > 0$ and $v_{eff}(y) = \int_x V(x,y)u_1(x,k_*)^2 \mathrm{d}x$.

Applied to $H^N - V$ in [Bruneau-Miranda-Raikov 14] and [Hislop-Soccorsi 14].

Results for the half-plane Landau Hamiltonian

In the half-plane model, no existence of an "effective mass"! But still:

Theorem [Bruneau-Miranda-Raikov 13]

In the case $H = H^D$, $\mathcal{N}(r)$ behaves like the number of negative eigenvalues of

$$\lambda_1^D - E_1 - \mathcal{V}$$

with $\mathcal{V}$ the anti-Wick symbol of V:

$$\mathcal{V} := \int_{x,\xi} V(x,\xi)\mathcal{P}_{x,\xi}\mathrm{d}x\mathrm{d}\xi$$

with

$$\mathcal{P}_{x,\xi} := |\psi_{x,\xi} >< \psi_{x,\xi}|, \quad \psi_{x,\xi}(k) := e^{ik\xi}e^{\frac{-(x-k)^2}{2}}$$

Corollary: for non-zero compactly supported potential V, $H^D - V$ has infinite discrete spectrum since $H^N - V$ has finite discrete spectrum.

3d model

An analogous 3d model: Schrödinger operator in $\mathbb{R}^3$ with magnetic field created by an infinite rectilinear wire bearing a constant current:

$$B(x,y,z) = \frac{1}{r^2}(-y,x,0), \quad r := \sqrt{x^2+y^2}$$

r : distance to the wire.

$$H_A := (-i\nabla - A)^2 = D_x^2 + D_y^2 + (D_z - \log r)^2,$$

Fibration:

- Cylindrical coordinates + angular FT Φ + z-FT $\mathcal{F}_3$:

$$\Phi \mathcal{F}_3 H_A \mathcal{F}_3^* \Phi^* := \sum_{m\in\mathbb{Z}}^{\oplus} \int_{k\in\mathbb{R}}^{\oplus} g_m(k) dk$$

$$g_m(k) := -\partial_r^2 + \frac{m^2 - \frac{1}{4}}{r^2} + (\log r - k)^2, \quad on\ L^2([0,+\infty), dr)$$

- $\sigma(g_m(k)) = \{\lambda_{n,m}(k),\ n\in\mathbb{N}^*, m\in\mathbb{Z}, k\in\mathbb{R}\}$.
- $\lambda'_{m,n}(k) < 0$ and $\lambda_{m,n}(k) \to 0$ as $k\to+\infty$. [Yafaev 03, Yafaev 08].

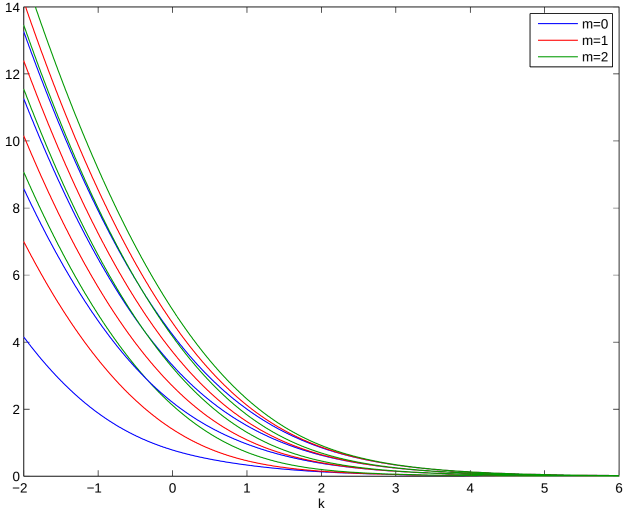
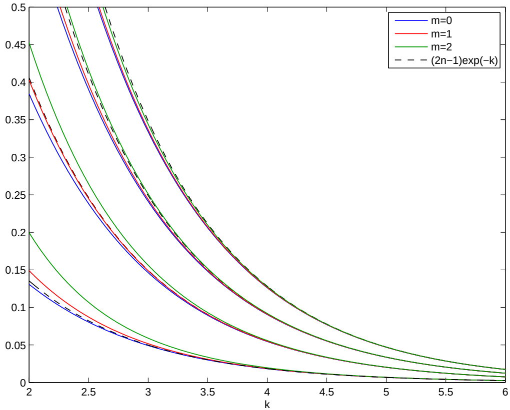


Figure: Band functions for a magnetic field created by an infinite wire.

Theorem (B.-Popoff '14)

For all $(m, n) \in \mathbb{Z} \times \mathbb{N}^*$, there exist constants $C_{m,n} > 0$ and $k_0 \in \mathbb{R}$ such that

$$\forall k \geq k_0, \quad |\lambda_{m,n}(k) - (2n-1)e^{-k} + (m^2 - \frac{1}{4} - \frac{n(n-1)}{2})e^{-2k}| \leq C_{m,n} e^{-5k/2}$$



Results for suitable perturbations

Theorem (Bruneau-P. '14)

Assume $V(r,z) \geq \langle r \rangle^{-\alpha} v_{\perp}(z), \qquad \alpha > 0$ with
(i) $\alpha < \frac{1}{2}$ and $v_{\perp} \in L^1(\mathbb{R})$ such that $\int_{\mathbb{R}} v_{\perp}(z)dz > 0$.
or
(ii) $v_{\perp} \geq C\langle z \rangle^{-\gamma}$ with $\gamma > 0$ and $\alpha + \frac{\gamma}{2} < 1$
Then $H_A - V$ have a infinite number of negative eigenvalues.

Theorem (Bruneau-P. '14)

Assume V is a relatively compact perturbation of H_A such that
$V(x,y,z) \leq \langle (x,y) \rangle^{-\alpha} v_{\perp}(z)$, with $\alpha > 1$ and $0 \leq v_{\perp} \in L^1(\mathbb{R})$.
Then $H_A - V$ have, at most, a finite number of negative eigenvalues.

Note that in some case, $H_A - V$ have finite discrete spectrum since $-\Delta - V$ has not!

Plan

Limiting absorption principle (LAP)

LAP for fibered operator

- Outside the thresholds: Well known LAP on weighted spaces

$$L^{2,s} := \{\varphi \in L^2, (1+y^2)^{s/2}\varphi \in L^2\}.$$

- On non-degenerate critical point $\lambda_n'(k_*) = 0$, LAP on the NH^s spaces [Croc-Dermenjian 95]

$$NH^s := \{\varphi \in L^{2,s}, \varphi_n(k_*) = 0\}.$$

- Extended to any critical point ("reached thresholds") in [Soccorsi 99].

New frame work for non-reached thresholds:

- LAP for the Landau Dirichlet Hamiltonian in a half-plane [P-Soccorsi 15] at Landau level.
- The absorption spaces require exponential decay on $k \mapsto \varphi_n(k)$.
- Associate functions are localized near the edge though phase-space analysis.

This general method is easily adapted to any non-reached threshold!

To go further

Expansion of the resolvent

- Expansion of the resolvent at reached threshold (critical point of the band functions) is done in general in [Gérard 90].
- Description of the resonances for small electrical perturbations follows [Grigis-Klopp 95]. You need to know some algebraic geometry to extend singular integral on the universal covering of the resolvent set.
- Non-reached thresholds needs a new framework and will give rise to new singularity of the resolvent.
  Example of the Landau Hamiltonian in a half-plane:
  $\lambda_n(k) - E_n \underset{k\to+\infty}{\sim} k^{2n-1}e^{-k^2} \implies$ essential singularity of the symbol at ∞.
- Global expansion: complexify the frequency k.

And perturbations

- Once you have extended the resolvent around the thresholds, study eigenvalues and resonances for perturbations (electric potential, deformation of the boundary, obstacles...).
- Numerics will help!!

Iwatsuka's magnetic steps

Piecewise constant magnetic field

- Snakes'orbit in 2D electron gas [Peeter-Reijniers 2000].
- Magnetic field in $\mathbb{R}^2$:

$$B(x,y) = B(x) = \begin{cases} b_1 & \text{if} \quad x < 0 \\ b_2 & \text{if} \quad x > 0 \end{cases}$$

- When $0 < b_1 < b_2$, the band function are increasing [Hislop-Soccorsi 13].
- $b_1 = 0 \implies \mathfrak{h}(k)$ does not have compact resolvent: special treatment.
- By scaling, we are reduced to $b_1 < 0 <$ and $b_2 = 1$.

Limit for large frequencies:

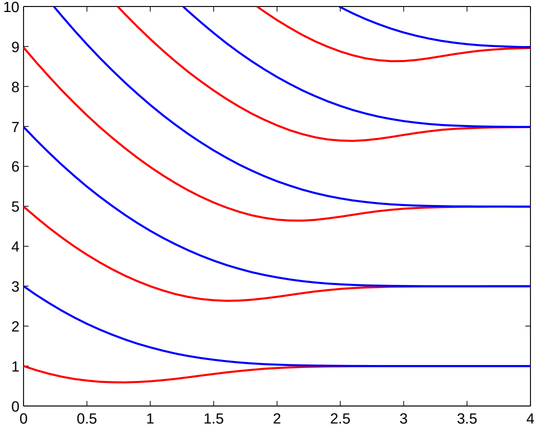
- $\lambda_n(k) \to +\infty$ as $k \to -\infty$. Let $\mathcal{L}$ be the set of the Landau levels:

$$\text{As } k \to +\infty, \text{the limits of the band functions is precisely} \quad \mathcal{T} := \mathcal{L} \cup |b_1|\mathcal{L}.$$

- The dispersion curves depend on $r := \frac{b_1}{b_2}$.
- Disjonction depending on wether r has the form $-\frac{2n-1}{2m-1}$ or not.

Courbes de dispersions du modèle d'Iwatsuka

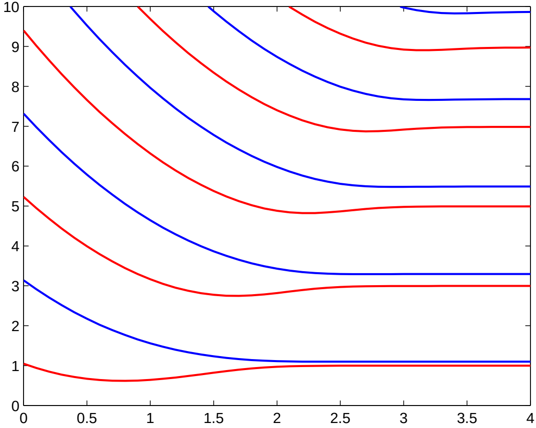
$$b_1 = -1, \quad b_2 = 1, \quad r = -\frac{1}{1}$$



En abscisse : k paramètre de Fourier dual de y.

Courbes de dispersions du modèle d'Iwatsuka

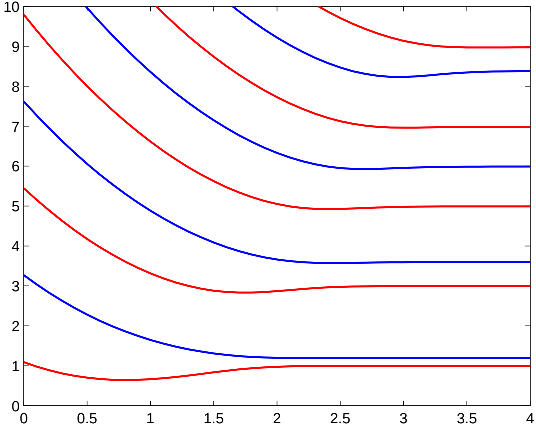
$$b_1 = -1.1, \quad b_2 = 1, \quad r = -\frac{11}{10}$$



En abscisse : k paramètre de Fourier dual de y.

Courbes de dispersions du modèle d'Iwatsuka

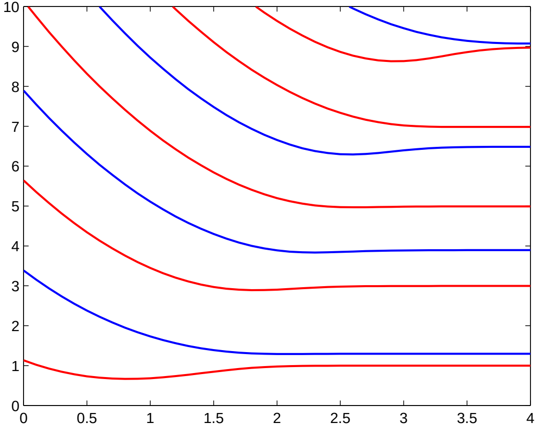
$$b_1 = -1.2, \quad b_2 = 1, \quad r = -\frac{6}{5}$$



En abscisse : k paramètre de Fourier dual de y.

Courbes de dispersions du modèle d'Iwatsuka

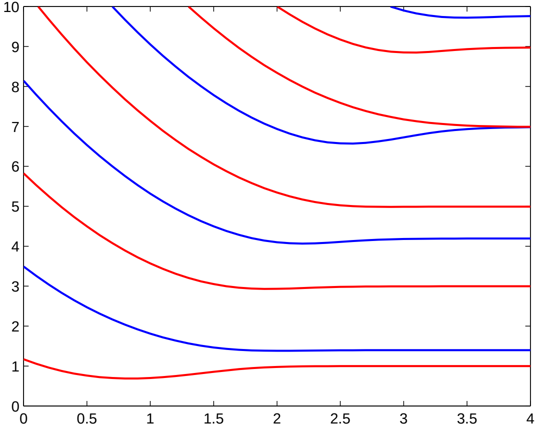
$$b_1 = -1.3, \quad b_2 = 1, \quad r = -\frac{13}{10}$$



En abscisse : k paramètre de Fourier dual de y.

Courbes de dispersions du modèle d'Iwatsuka

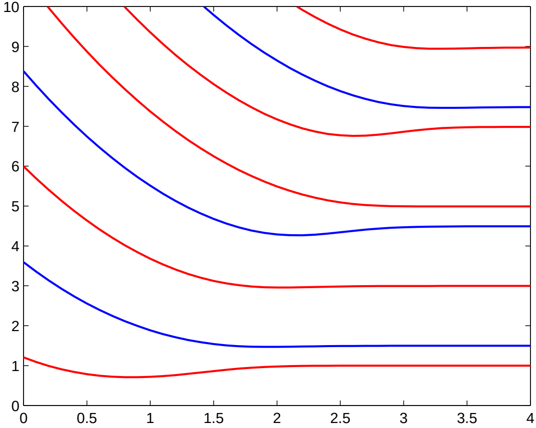
$$b_1 = -1.4, \quad b_2 = 1, \quad r = -\frac{7}{5}$$



En abscisse : k paramètre de Fourier dual de y.

Courbes de dispersions du modèle d'Iwatsuka

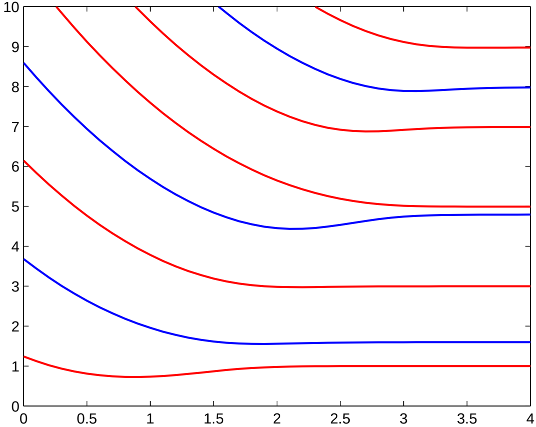
$$b_1 = -1.5, \quad b_2 = 1, \quad r = -\frac{3}{2}$$



En abscisse : k paramètre de Fourier dual de y.

Courbes de dispersions du modèle d'Iwatsuka

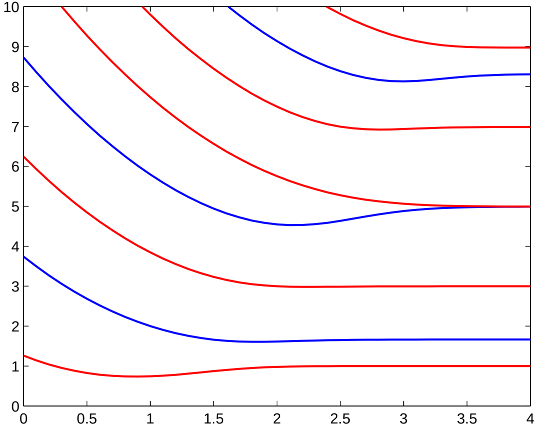
$$b_1 = -1.6, \quad b_2 = 1, \quad r = -\frac{8}{5}$$



En abscisse : k paramètre de Fourier dual de y.

Courbes de dispersions du modèle d'Iwatsuka

$$b_1 = -1.667, \quad b_2 = 1, \quad r = -\frac{5}{3}$$



En abscisse : k paramètre de Fourier dual de y.

Asymptotics of the band function

Theorem [Hislop-Persson-Popoff-Raymond 16]

1. (Non splitting case). Let $E \in \mathcal{T} \setminus (\mathcal{L} \cap |b_1|\mathcal{L})$. Then there exists a unique band function converging to E, and the convergence is by above:

$$\lambda_{p_n}(k) = \begin{cases} E - C_n k^{2n-3} e^{-k^2}(1+o(1)) & \text{if } E = 2n-1 \\ E - \widetilde{C}_n k^{2n-3} e^{-k^2/|b_1|}(1+o(1)) & \text{if } E = |b_1|(2n-1) \end{cases}$$

2. (Splitting case). Let $E \in \mathcal{L} \cap |b_1|\mathcal{L}$. Write $E = E_n = |b_1|E_m$. Two band functions converge toward E, one from below and one from above. The threshold is asymptotically degenerate and

$$\begin{cases} \lambda_{p_n}(k) = E - C_n k^{2n-3} e^{-k^2}(1+o(1)) \\ \lambda_{p_n+1}(k) = E + \widetilde{C}_m k^{2m+1} e^{-k^2/b}(1+o(1)) \end{cases}$$

Method and comments

Consequences:

- When $|b_1|$ has not the form $-\frac{2n-1}{2m-1}$, all the band functions have a global minimum. The quantum transport can occur in two opposite directions since λ_n are not monotonous.
- Precise asymptotics is needed for precise description of the threshold.

Method

- The large k limit is equivalent to a 1d semi-classical problem with double wells:

$$\mathfrak{h}(k) \equiv k^2 \left(k^{-4} D_X^2 + V(X)\right), \quad V(X) = \begin{cases} (xb_1 - 1)^2, & x < 0 \\ (x-1), & x > 0 \end{cases}$$

- Better to use special functions: λ is an eigenvalue of $\mathfrak{h}(k)$ iff

$$U(-\tfrac{\lambda}{2b}, -\sqrt{2}\tfrac{k}{\sqrt{b}})U'(-\tfrac{\lambda}{2}, -\sqrt{2}k) + \sqrt{b}U'(-\tfrac{\lambda}{2b}, -\sqrt{2}\tfrac{k}{\sqrt{b}})U(-\tfrac{\lambda}{2}, -\sqrt{2}k) = 0$$

where U is the first Weber parabolic cylindrical function.