Total absorption in waveguides by admittance conjugation

V. Pagneux

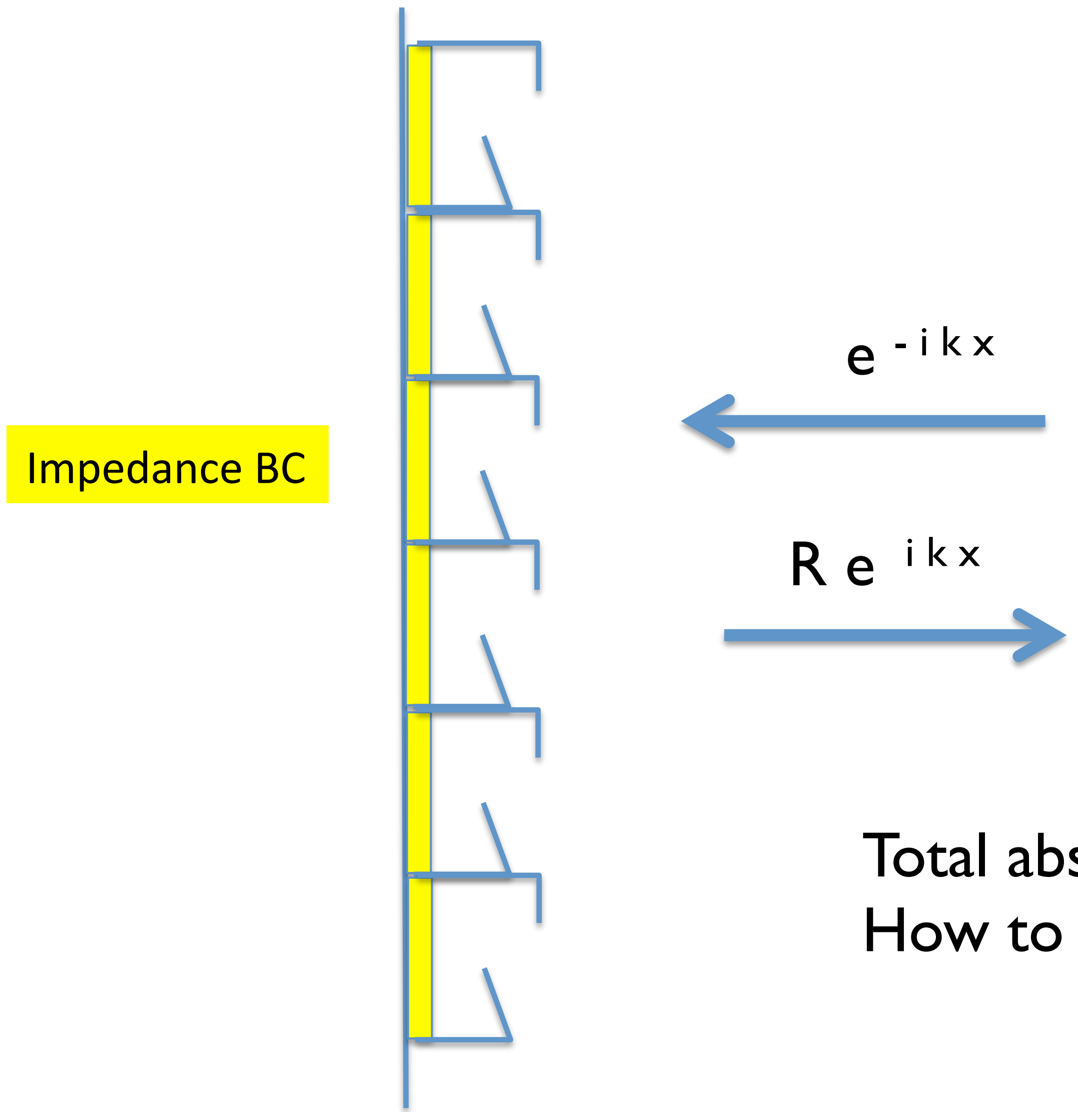
Lab. Acous. Univ. Maine – CNRS

Le Mans - France

CONFERENCE ON WAVEGUIDES, Porquerolles

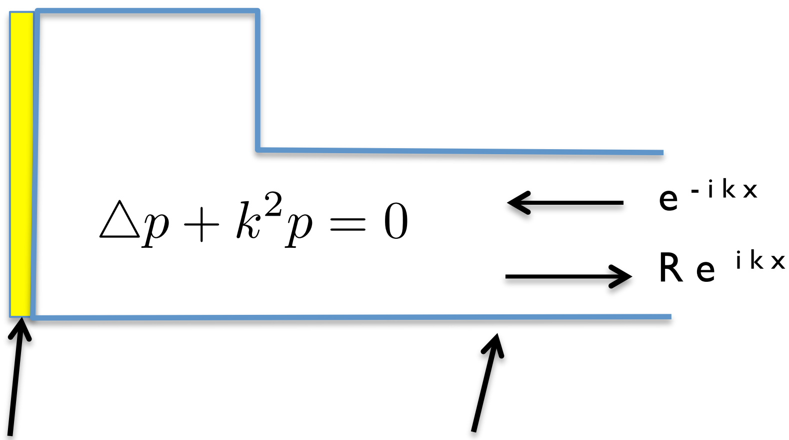
17th to 19th may 2016

Wall with total absorption



Total absorption when R = 0 .
How to choose the impedance ?

Will consider waveguide :
Reflection problem (acoustics)



$\partial_n p = A\,p$
(admittance BC
alias Robin BC
alias Mixed BC)

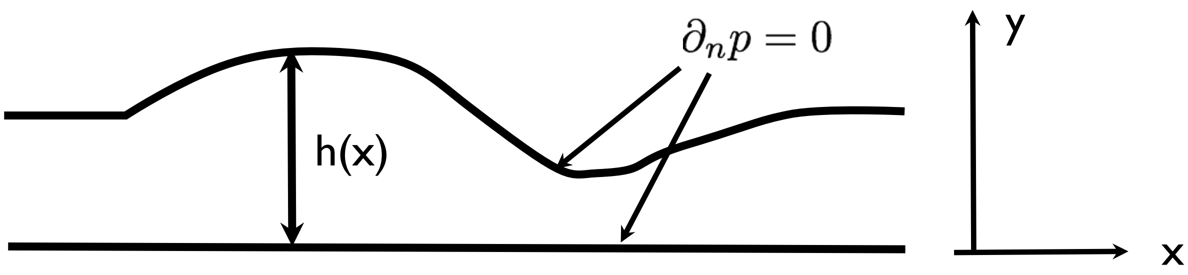
$\partial_n p = 0$
(Neumann BC)

Absorption by admittance at the wall:
Im (A) < 0 (convention $e^{-i\omega t}$)

- Looking for A values such that R=0 (total absorption)
- Difficult at low frequency : for small k, A has to be small (|A| << k for thin coating)
- Surprising connection with water waves

coupled mode equations

2D acoustics



evolution equation

$$\partial_x^2 p + \partial_y^2 p + k^2 p = 0 \quad \Longrightarrow \quad \partial_x \begin{pmatrix} p \\ \partial_x p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k^2 - \partial_y^2 & 0 \end{pmatrix} \begin{pmatrix} p \\ \partial_x p \end{pmatrix}$$

modal expansion

$$p(x,y) = \sum_{n \geq 0} a_n(x) g_n(y;x)$$

$$\partial_x p(x,y) = \sum_{n \geq 0} b_n(x) g_n(y;x)$$

local modes (Neumann)

$$g_n(y;x) = \sqrt{\frac{2-\delta_{n0}}{h}} \cos(n\pi y/h)$$

coupled mode equations:

$$a' = -Fa + b$$

$$b' = -K^2 a + F^T b$$

Unstable evolution eqs.

Riccati equation

admittance matrix
(~ DtN operator)

$$b = Y a$$

(b ← $\partial_x p$, a ← p)

from coupled mode eqs.

$$a' = -Fa + b$$
$$b' = -K^2 a + F^T b$$

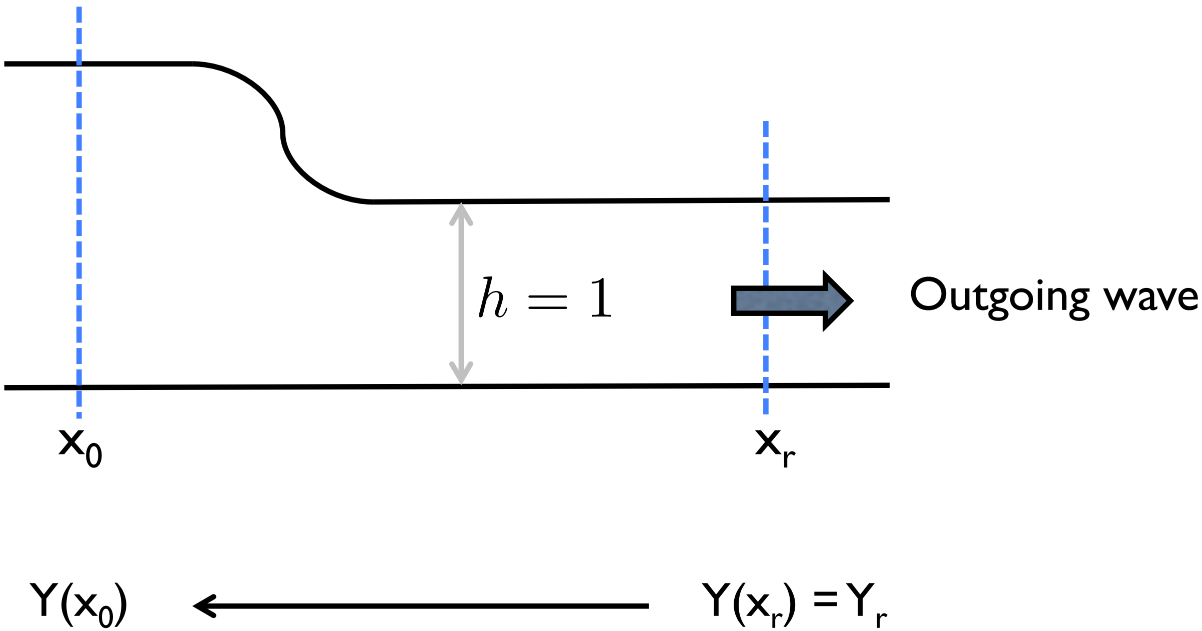
$$Y' = -K^2 - Y^2 + YF + F^T Y$$

Riccati equation

-initial condition : radiation condition (e.g. outgoing wave)
-numerically stable

-the same for the impedance matrix Z
-gives the scattering matrix (reflection & transmission)

Admittance computation



$Y(x_0)$ known from Y_r by the Riccati equation

$$Y' = -K^2 - Y^2 + YF + F^TY$$

Expression of the Y_r for outgoing waves

$$p = \sum_{n=0}^{\infty} a_n(x) g_n(y)$$

$$\partial_x p = \sum_{n=0}^{\infty} b_n(x) g_n(y)$$

$$a_n(x) = c_n e^{ik_n x} \longrightarrow b_n = ik_n a_n$$

Thus Y_r diagonal :

$$Y_r = \begin{pmatrix} ik_0 & & & \\ & ik_1 & & \\ & & ik_2 & \\ & & & \dots \end{pmatrix}$$

$$k_n = \sqrt{k^2 - n^2\pi^2}$$

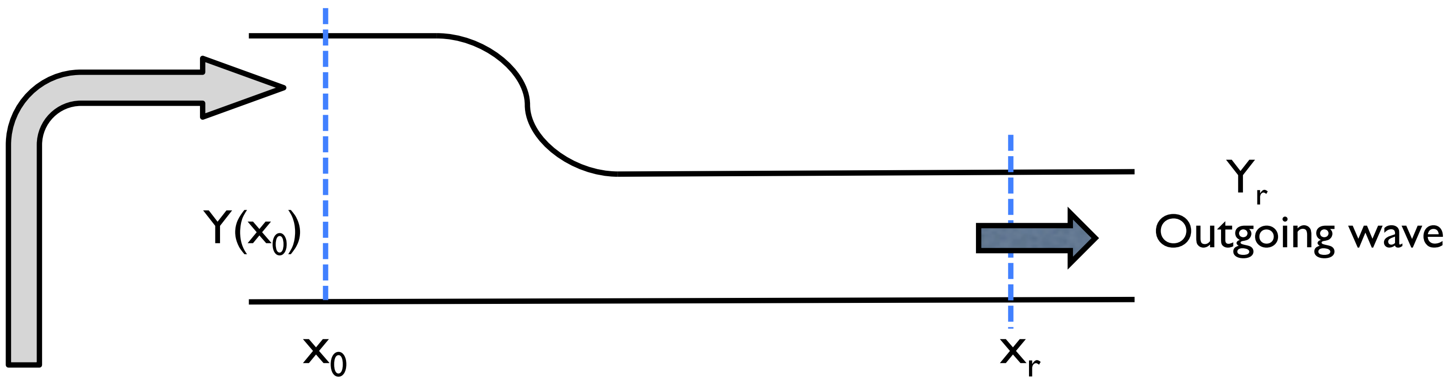
convention $e^{-i\omega t}$

k_n real (>0) : propagating

k_n pure imaginary (Imag >0) : evanescent

$k_0 = k$: mode n=0 always propagating

Eigenvalues of $Y(x_0)$

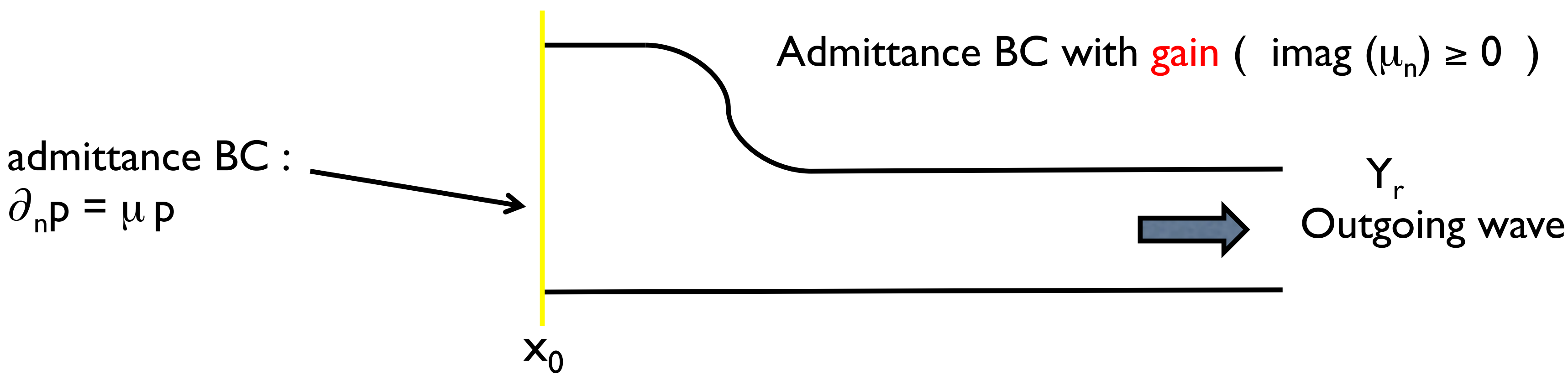


Eigenvalue problem $Y(x_0)\, u = \mu\, u$

eigenvalues μ_n and eigenvectors u_n
(imag $(\mu_n) \geq 0$ from radiation conds.)

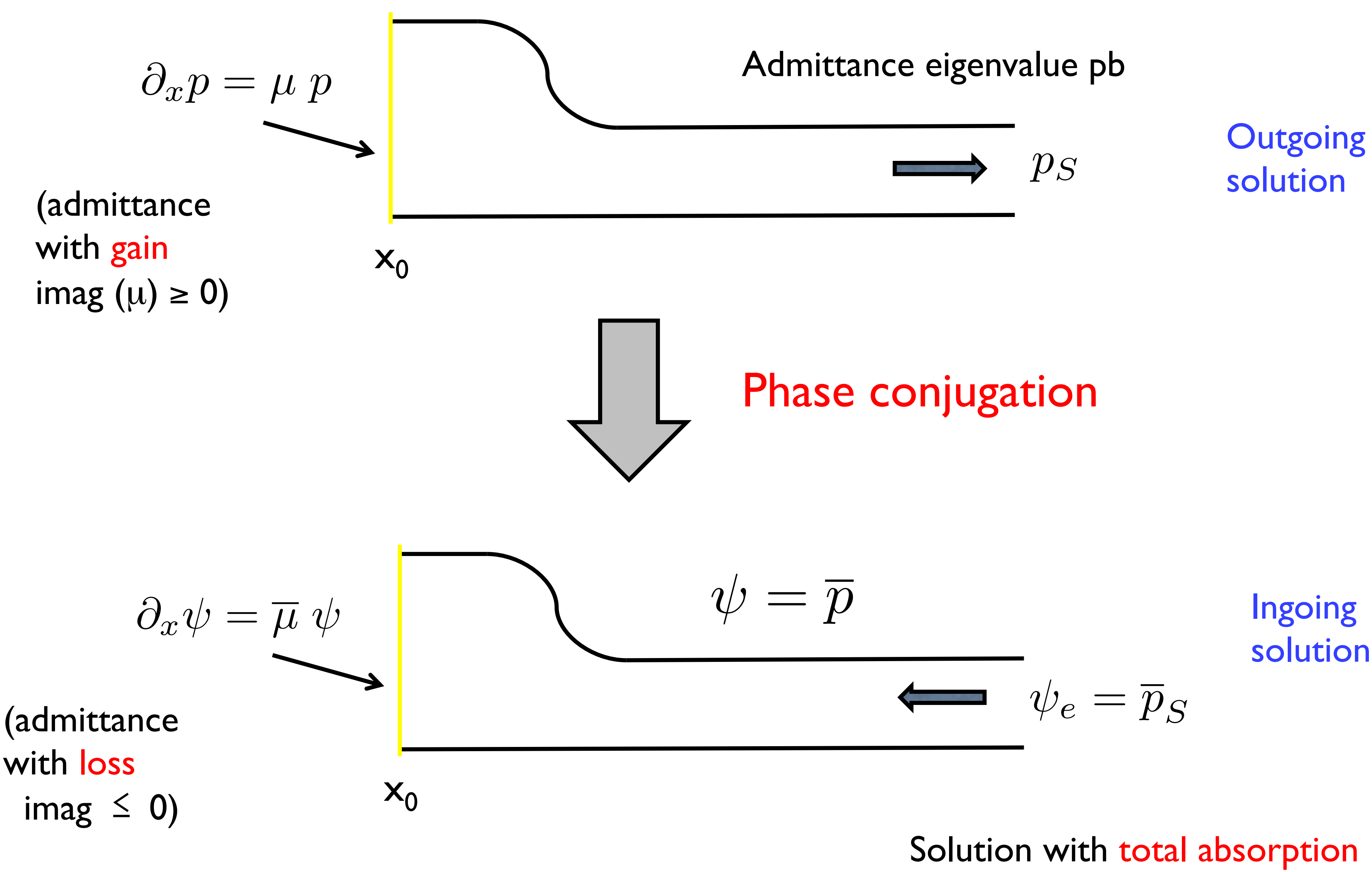
Solution associated to μ_n

Admittance BC with gain (imag $(\mu_n) \geq 0$)



Remark: if $\mu=0$ trapped mode is found

Admittance conjugation



Total absorption

In the following $k < \pi$: only the mode 0 is propagating

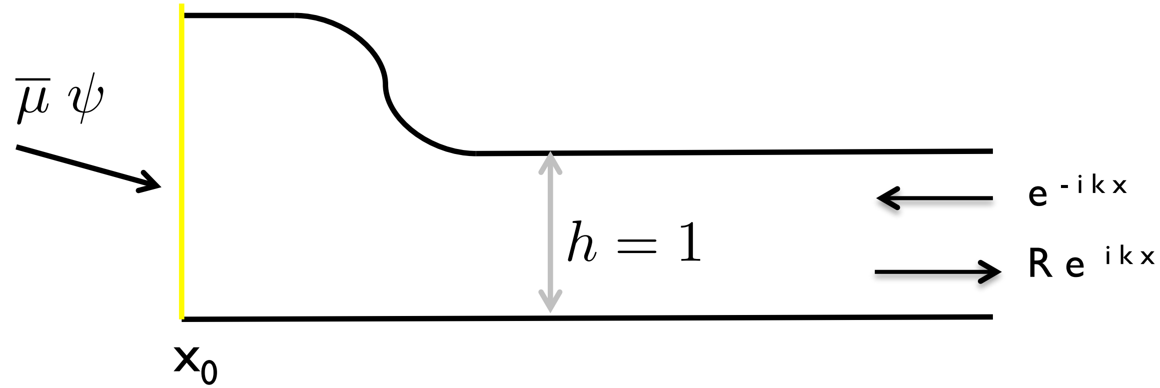
Absorption at low frequencies

$$Y_r = \begin{pmatrix} ik & & & \\ & -|k_1| & & \\ & & -|k_2| & \\ & & & \dots \end{pmatrix}$$

$$\partial_x \psi = \overline{\mu}\, \psi$$

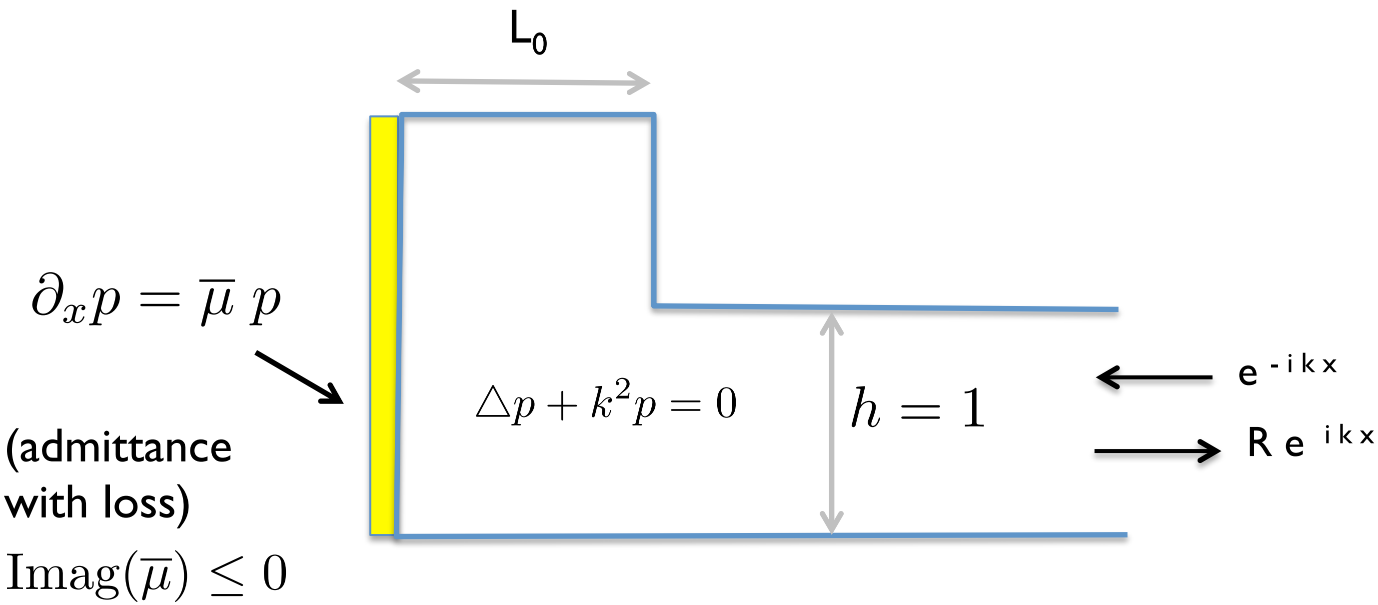
(admittance with loss)

$$\mathrm{Imag}(\overline{\mu}) \leq 0$$



To obtain total absorption R=0 :
choose the admittance μ as the eigenvalue problem of $Y(x_0)$ (outgoing wave pb)

Results

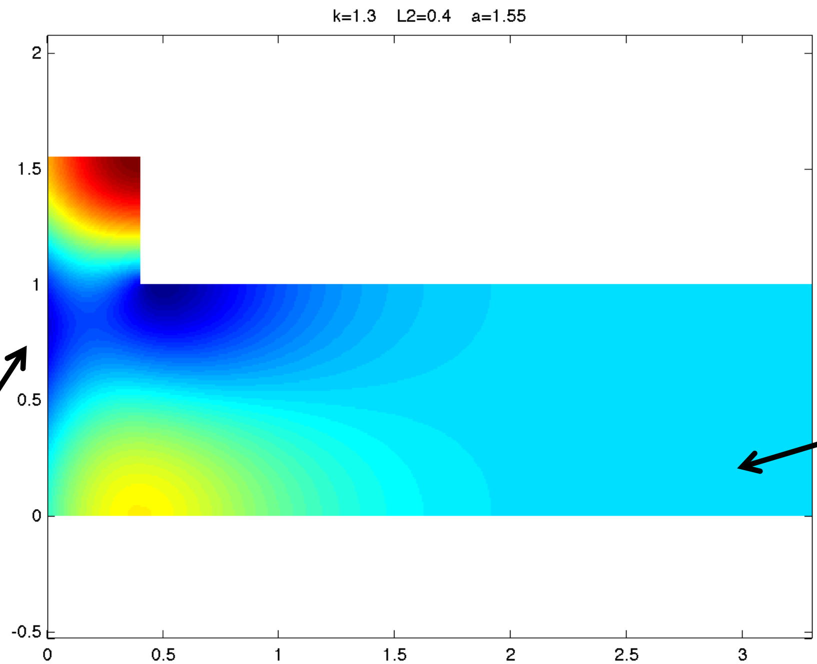


Where μ is one of the μ_0, μ_1, μ_2, … (eigenvalues of $Y(x_0)$ for outgoing waves)

Remark :
For a straight strip, $Y(x_0)=Y_r$ and only μ_0= ik gives an absorbing admittance.
This value (ik) is too large for thin coating of porous media at low freqs:
thus, we would like $|\mu| < k$

Total absorption example

| p(x,y) |



Mode matching

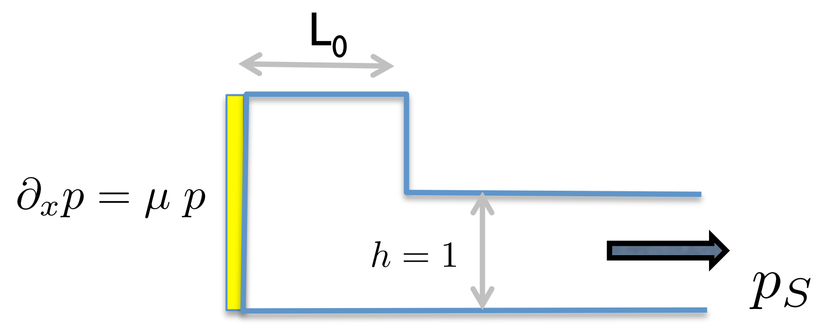
$$\partial_x p = \overline{\mu}\; p$$

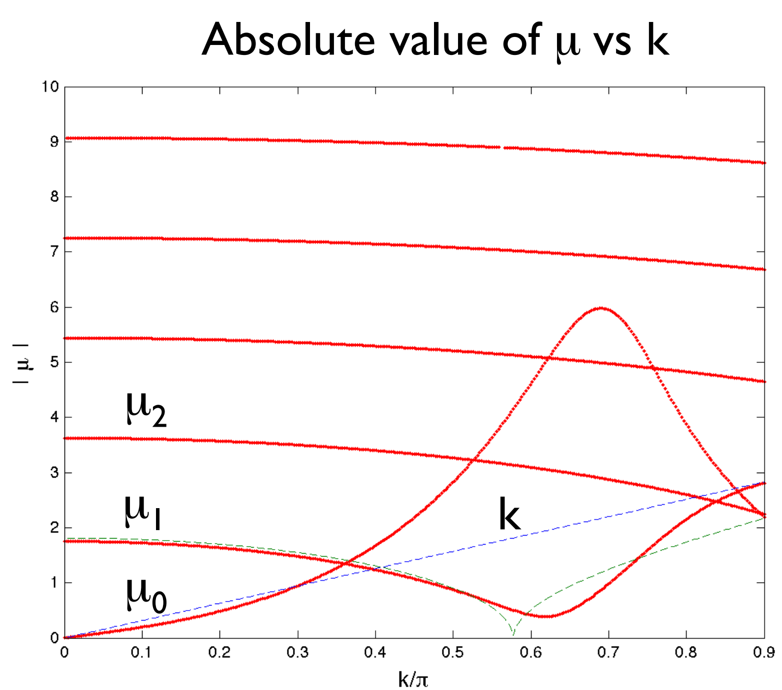
| p(x,y) | = 1
(constant since no reflection)

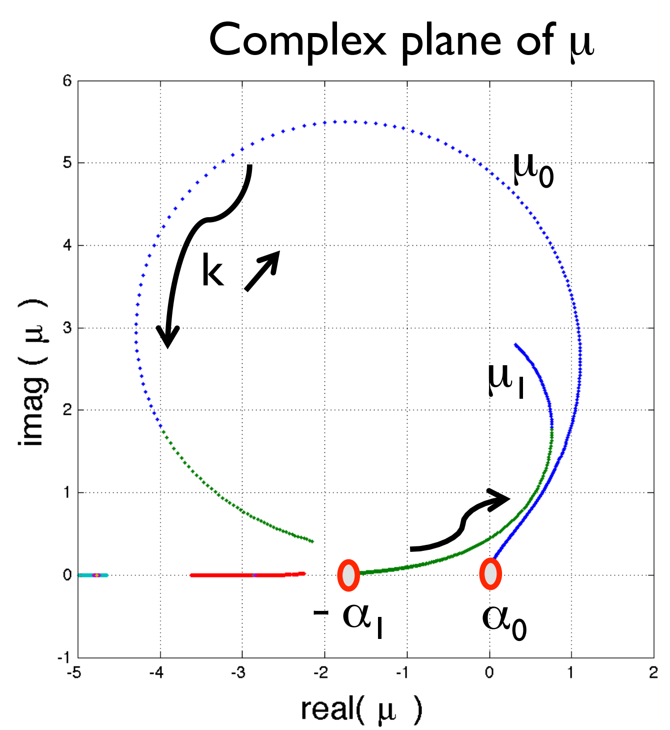
Finite element method

eigenvalues μ_n as a function of frequency k

$L_0 = 1$

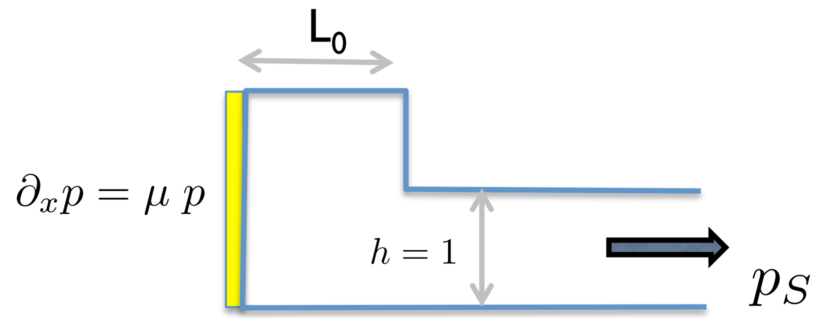






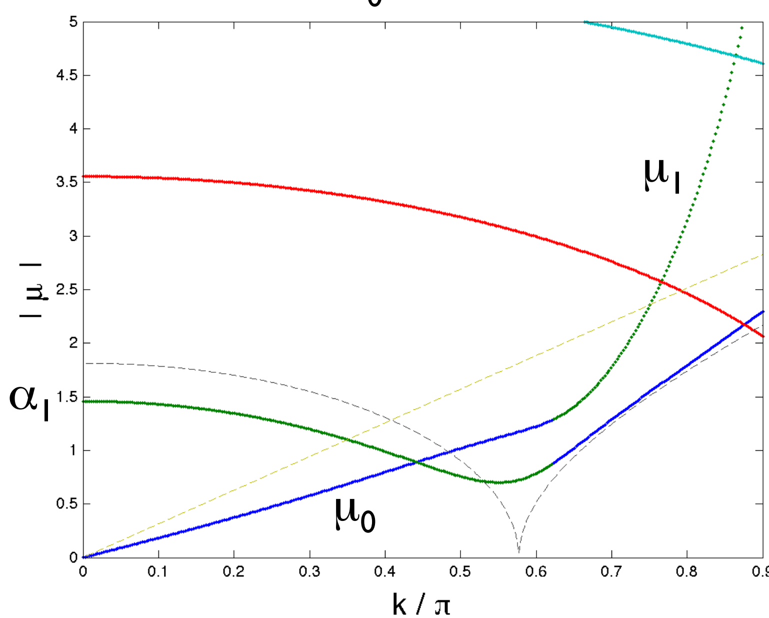
$-\alpha_n$: value of μ_n for $k \to 0$ ($\alpha_0 = 0$ and $\alpha_n > 0$)

Diminishing the length cavity L_0 : α_1 decreases

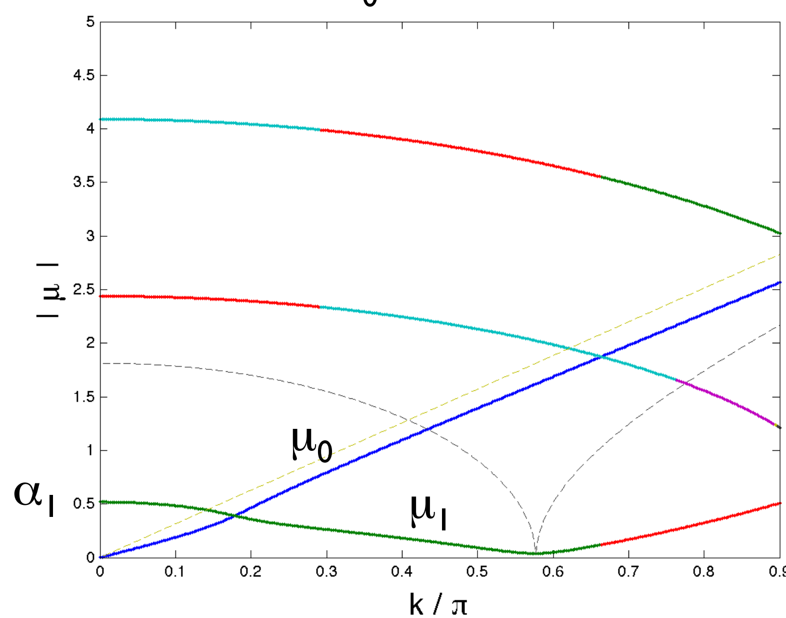


Absolute values of μ_n vs frequency k

$L_0 = 0.5$

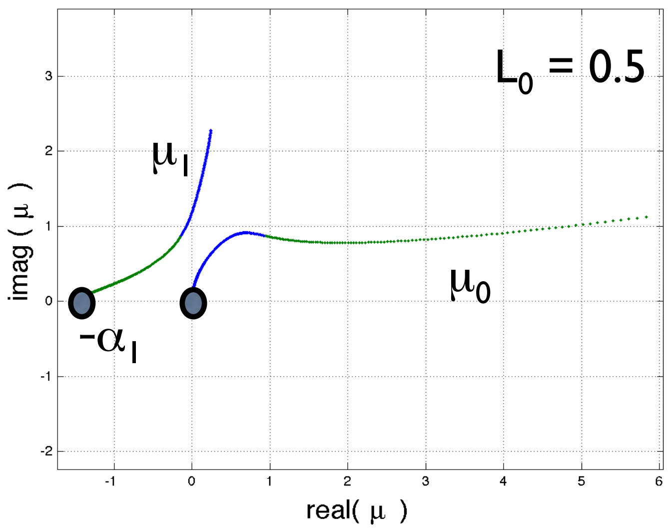


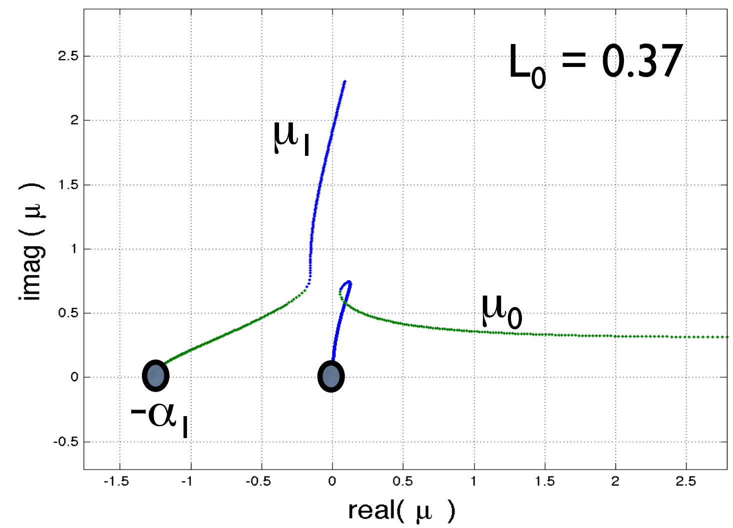
L_0=0.1

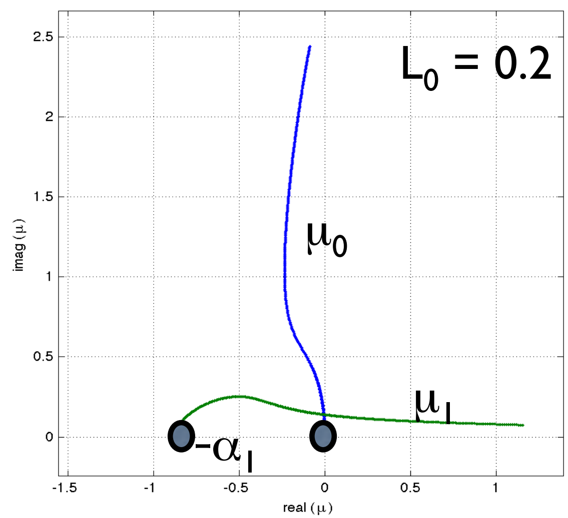


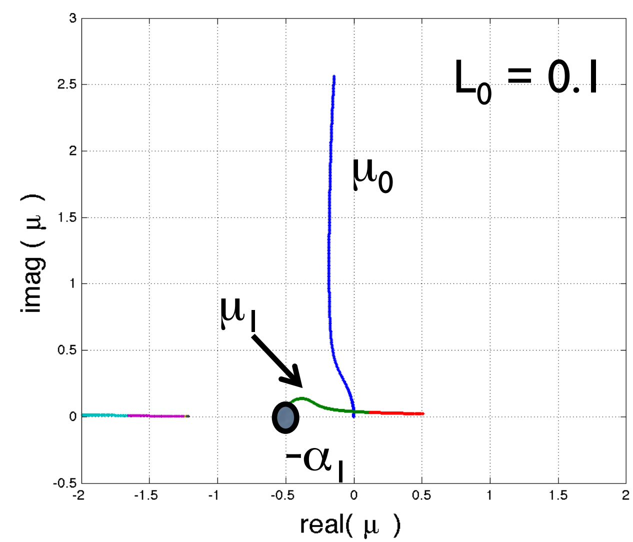
Possible : μ_n smaller than k

eigenvalues μ_n as a function of frequency k : complex μ plane



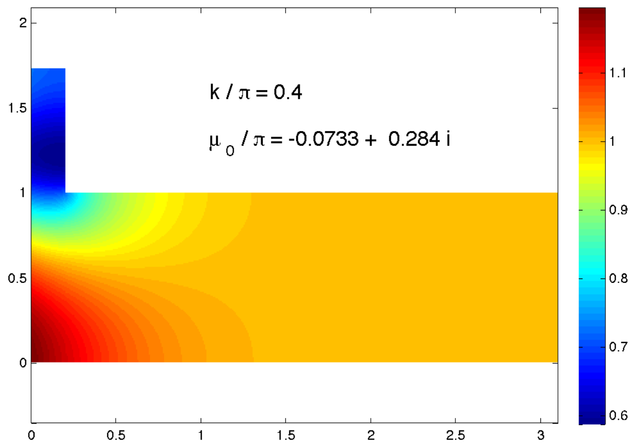




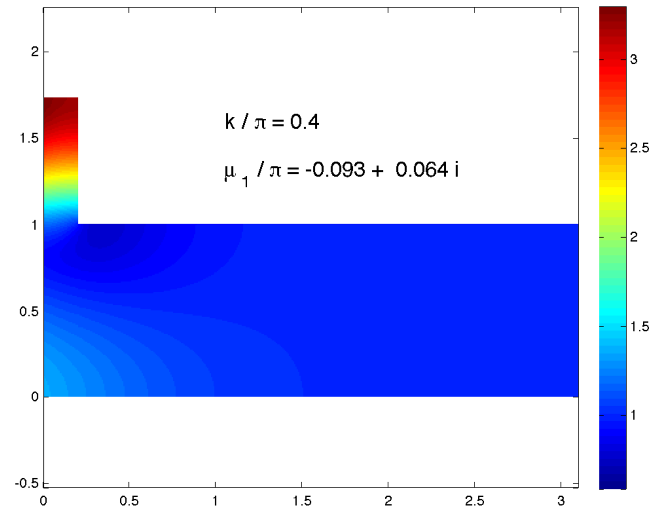


Different μ_n : different admittance BC for total absorption (for the same k)

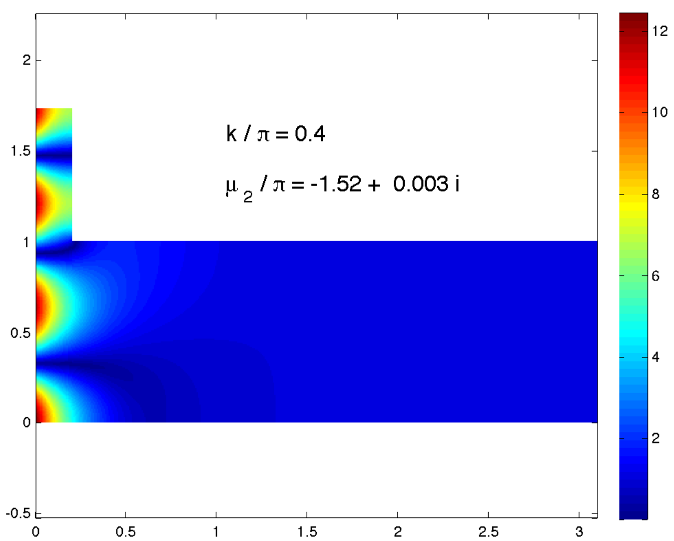
μ_0



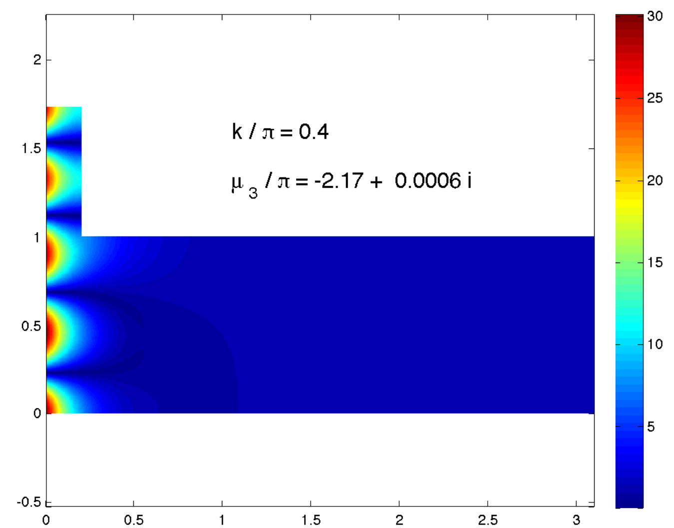
μ_1



μ_2



μ_3



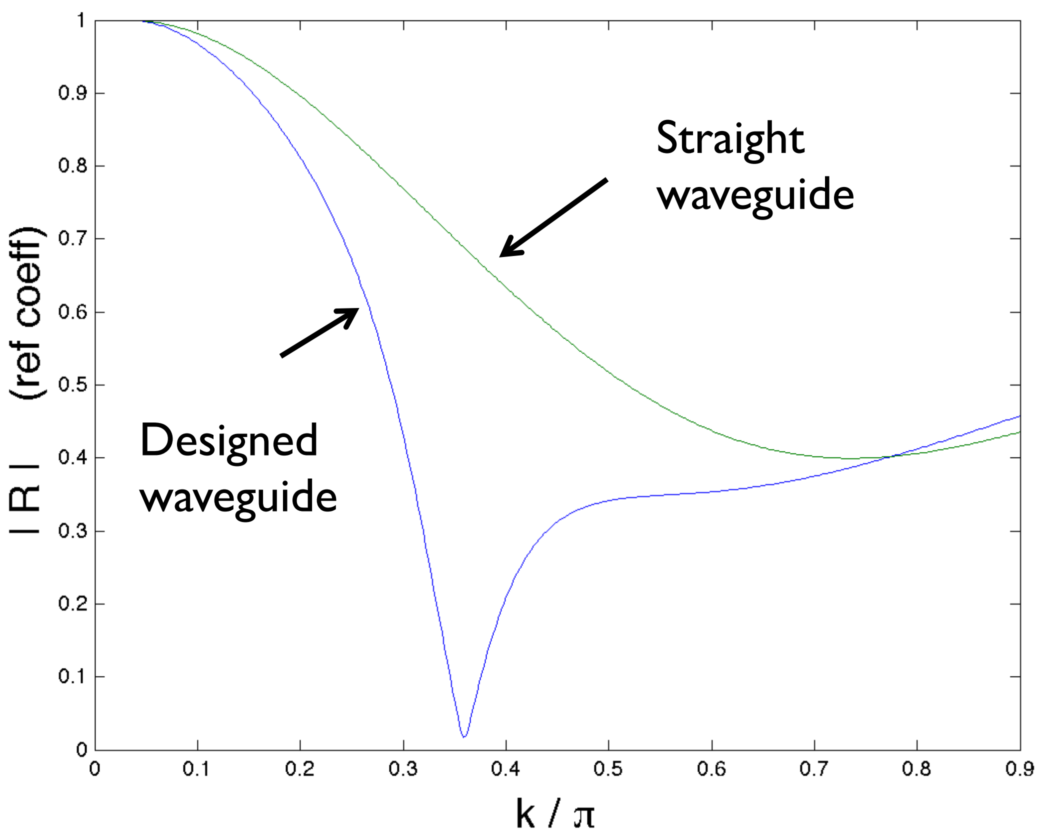
Reflection coefficient vs frequency k

same admittance

Comparison between

- Straight waveguide
- Heterogeneous waveguide designed to totally absorb at k=0.37 π

(admittance = thin porous coating similar to Delany-Basley)



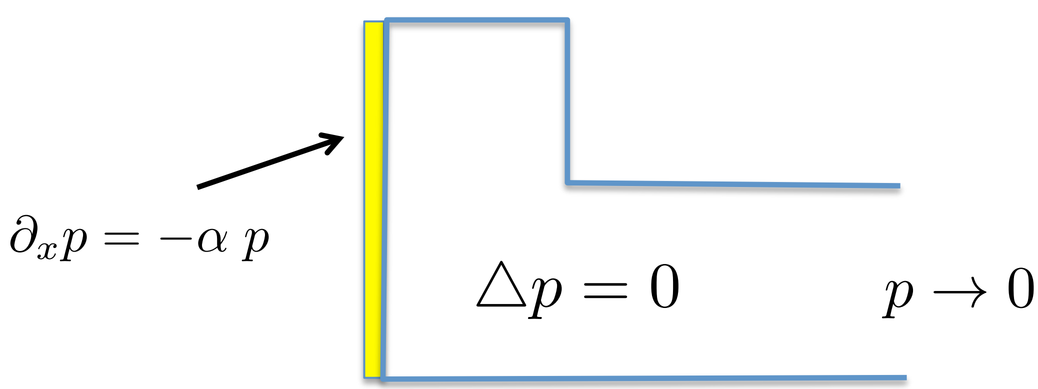
Connection with water waves

To get low values of μ_n at low frequency :

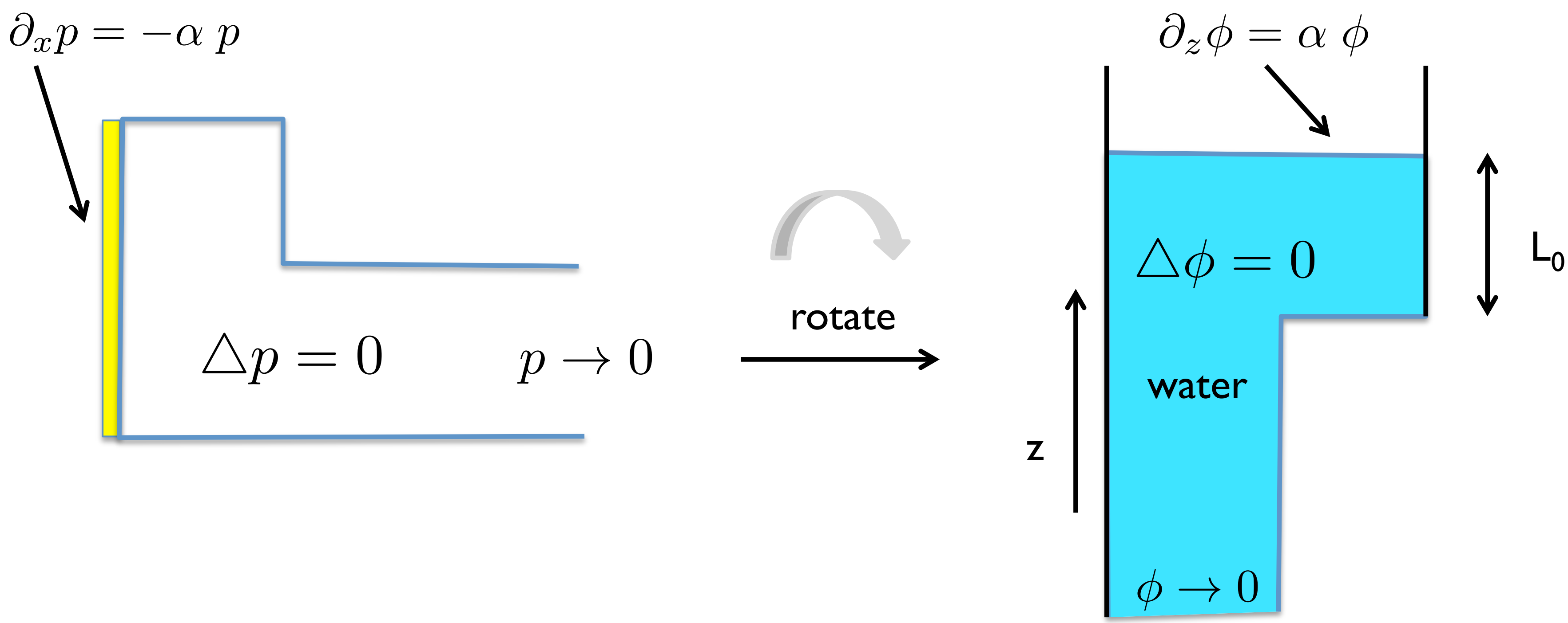
$\alpha_n = \lim \mu_n$ as $k \rightarrow 0$ is important

When α_n is small , total absorption by "small admittance" is possible at low frequency (i.e. absorption by thin structure)

Asymptotic expansion ($k \rightarrow 0$) : problem for α_n



Connection with water waves



α is the eigenvalue of the sloshing for water waves in cavity

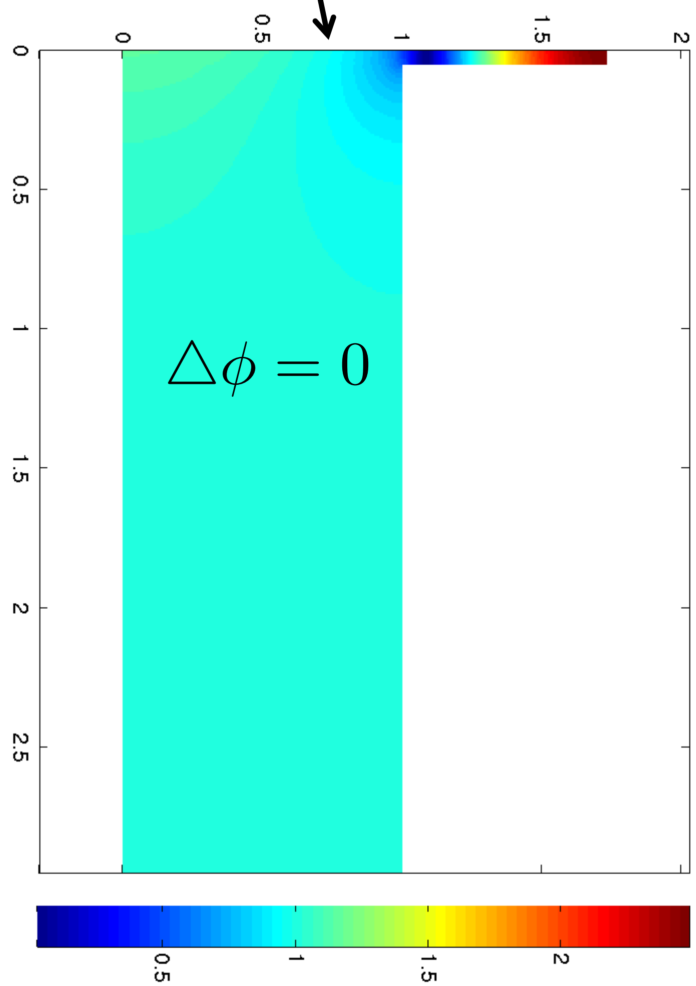
When $L_0 \to 0$: $\alpha_1 \to 0$ (shallow water has small wave speed)

$$\alpha_1 \simeq L_0 (\pi/d)^2$$

When $L_0 \rightarrow 0$: $\alpha_1 \rightarrow 0$ (shallow water has small wave speed)

$$\alpha_1 \simeq L_0(\pi/d)^2$$

$$\partial_z \phi = \alpha \, \phi$$



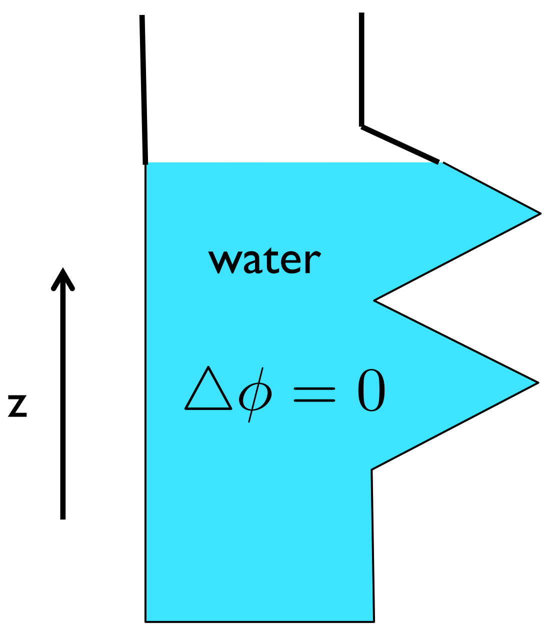
“flute mode”

Riccati for sloshing

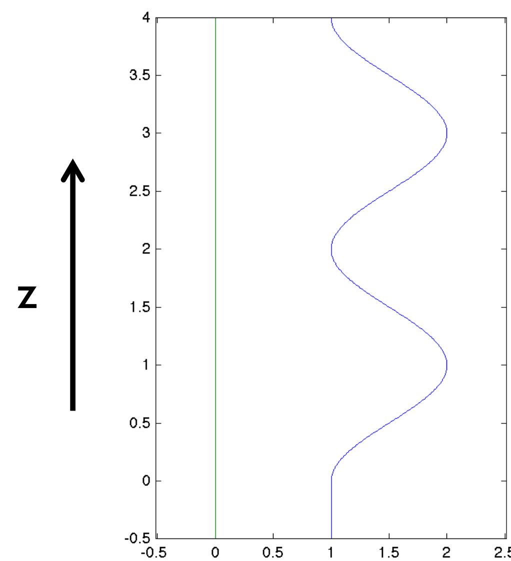
Integrating Riccati with z gives the sloshing frequency at each z (eigenvalues of Y(z))

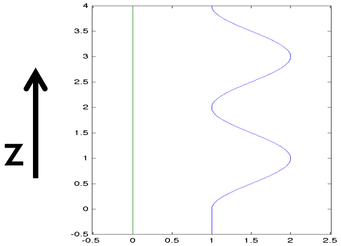
$$Y' = -\cancel{K^2} - Y^2 + YF + F^T Y$$

Riccati equation

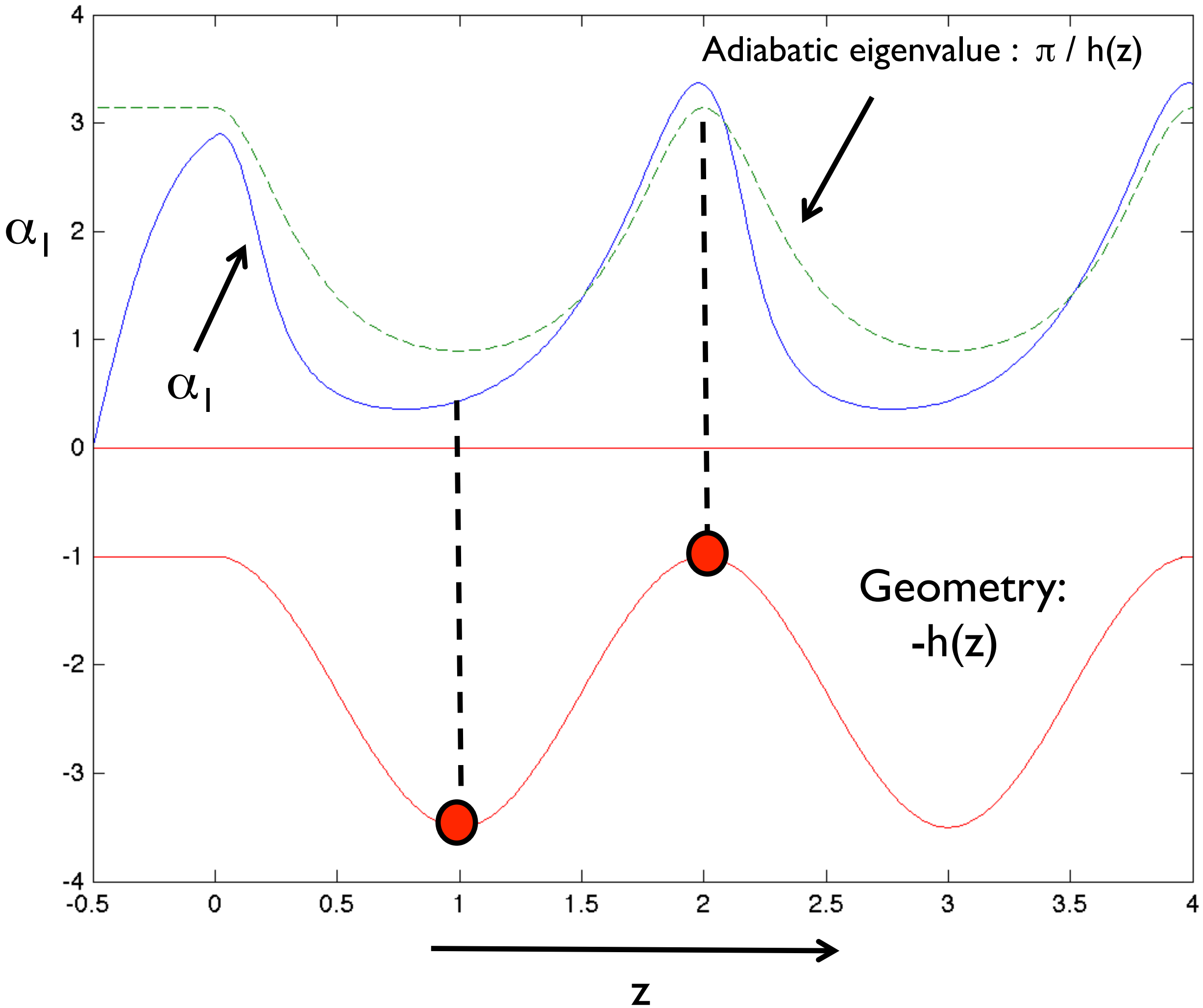


undulated glass





Shloshing eigenvalue α_1 as a function of z



Domain Monotonicity
for sloshing
at red points

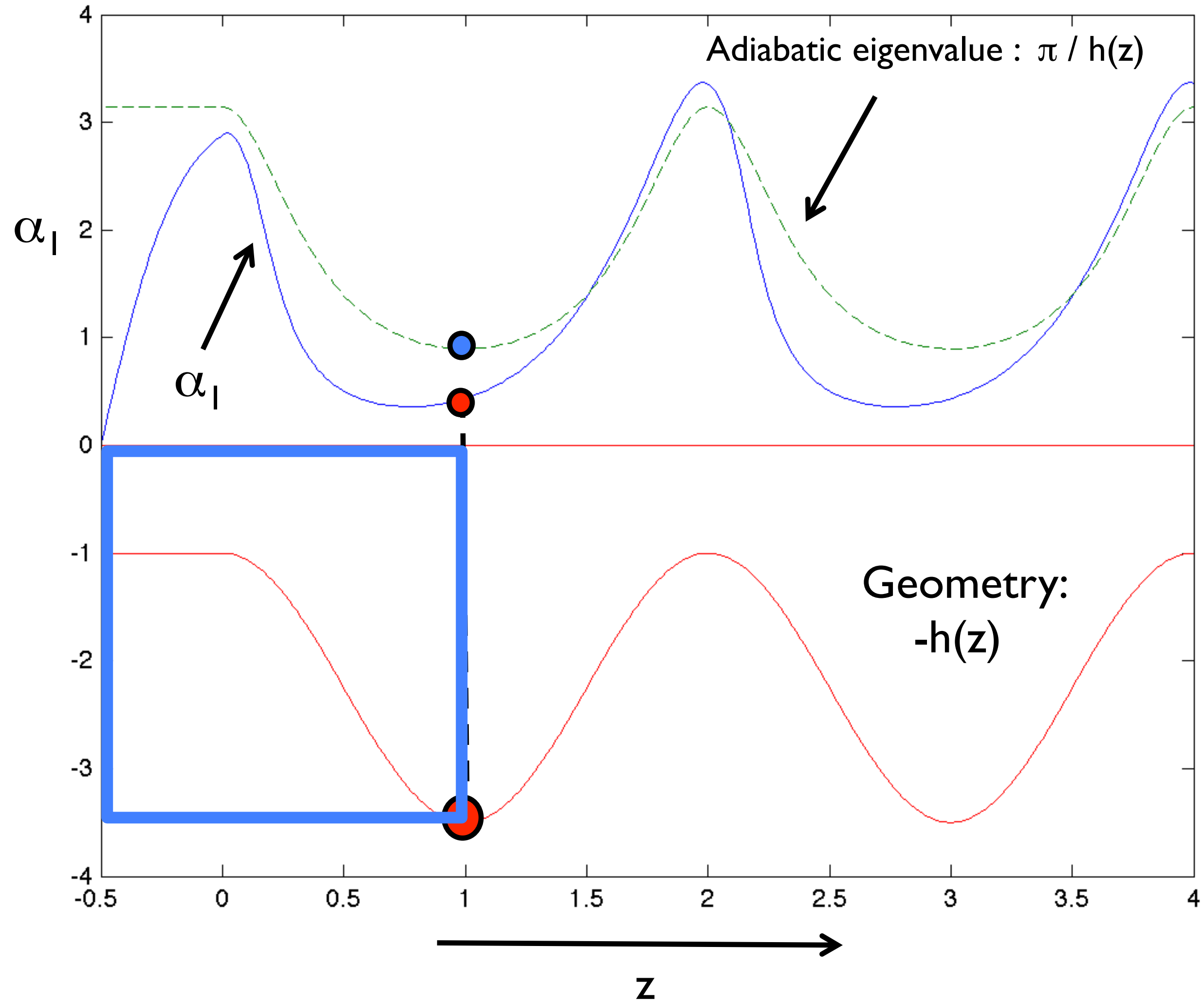
If $\hat{\Omega} \subset \Omega$

then $\hat{\alpha}_1 \leq \alpha_1$

(Ω volume of the cavity)

π/h : eigenvalue of rectangular cavity

$$\alpha_1 \leq \pi / h$$



Domain Monotonicity

for sloshing

at red points

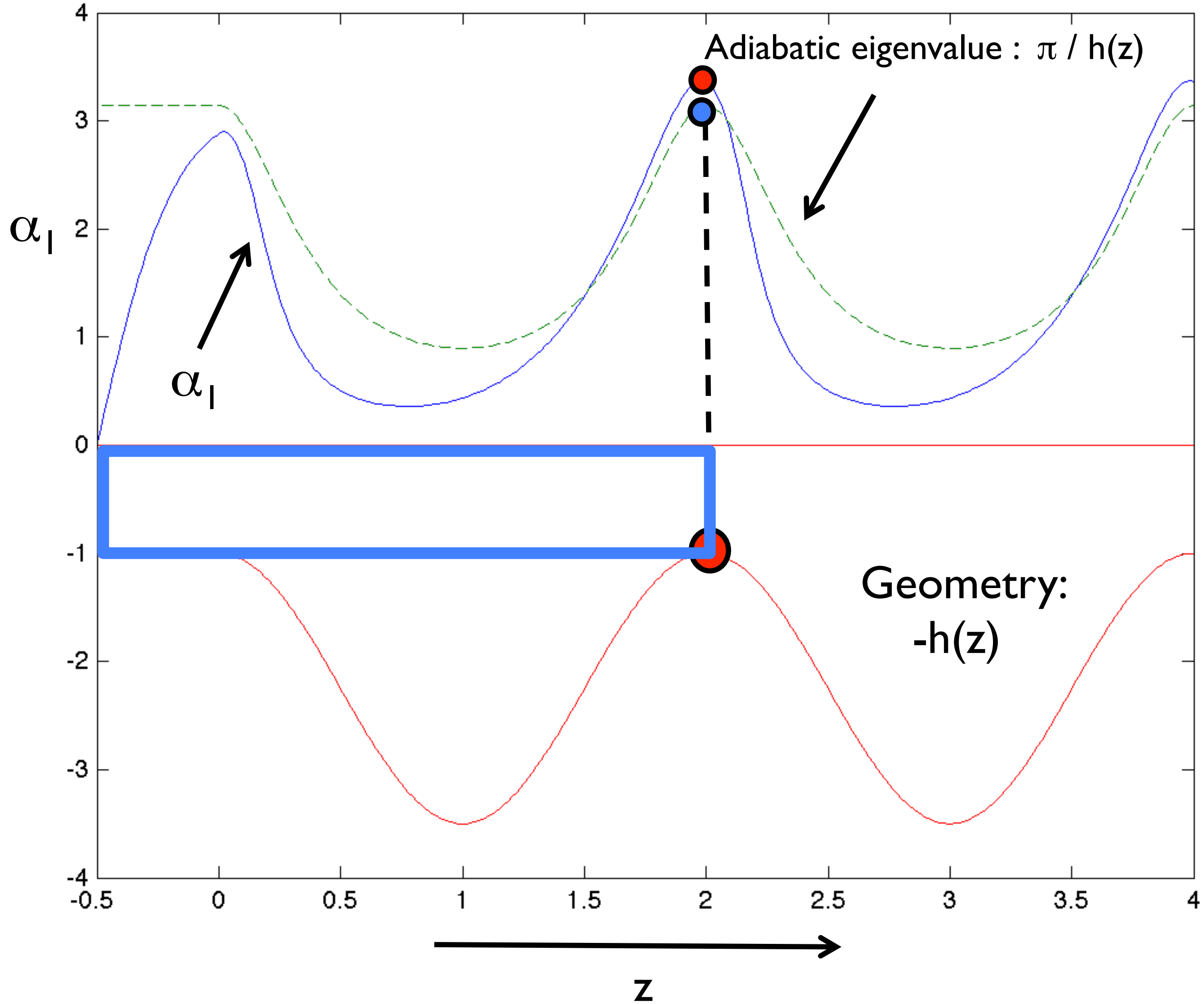
If $\hat{\Omega} \subset \Omega$

then $\hat{\alpha}_1 \leq \alpha_1$

(Ω volume of the cavity)

π/h : eigenvalue of rectangular cavity

$$\pi / h \leq \alpha_1$$



Domain Monotonicity
for sloshing
at red points

If $\hat{\Omega} \subset \Omega$

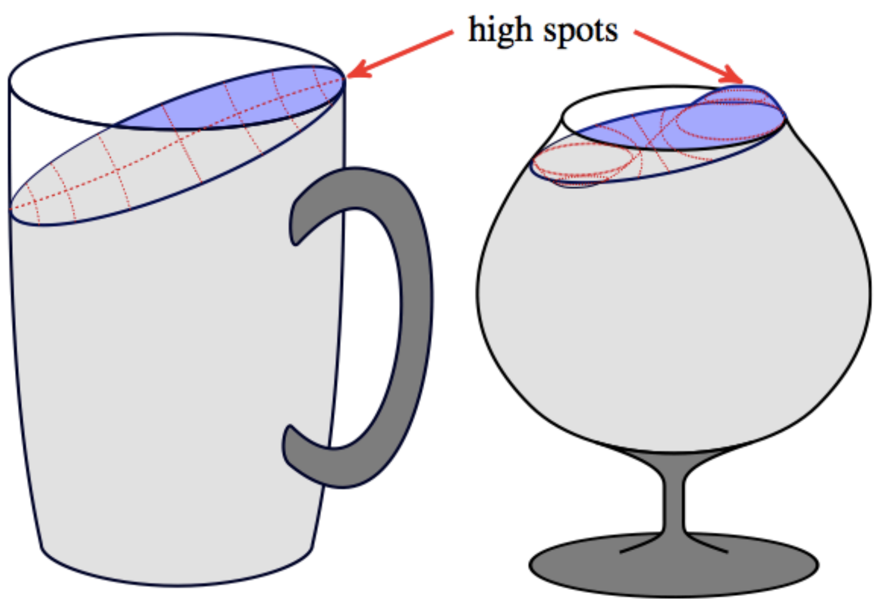
then $\hat{\alpha}_1 \leq \alpha_1$

(Ω volume of the cavity)

π/h : eigenvalue of rectangular cavity

Links to high spot (hot spot) problems:

Importance of the angle between the free surface and the wall



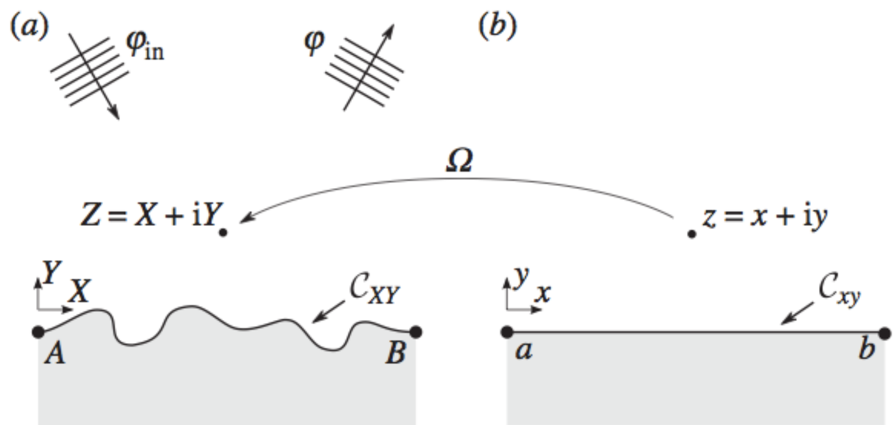
(A) A cup. (B) A snifter.

FIGURE 1. High spots in a coffee cup and a snifter.

Kulczycki, T., Kwasnicki, M., & Siudeja, B. (2013). Spilling from a cognac glass. *arXiv preprint arXiv:1311.7296*.

Remark (rough surface)

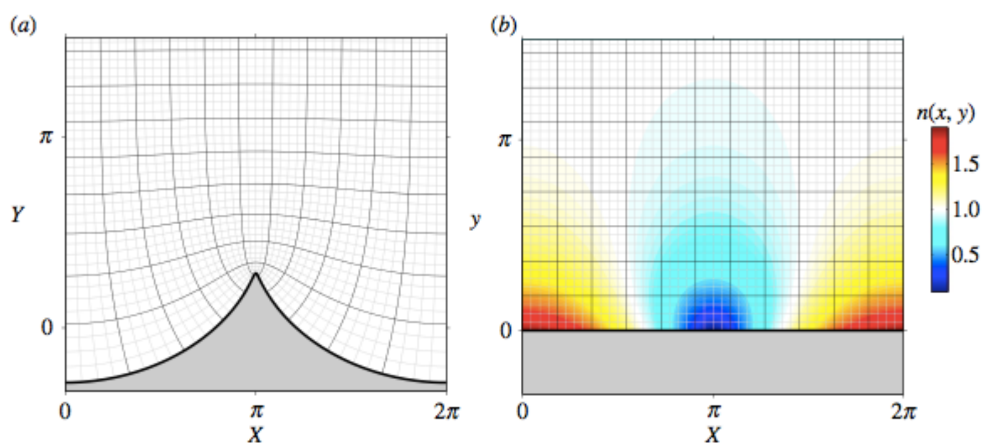
The same (multimodal Riccati) can be done for rough surface by conformal mapping



$$\Delta_{XY}\Phi + k^2\Phi = 0 \qquad \Delta_{xy}\Phi + k^2 n^2 \Phi = 0$$

“Refractive index”

$$n(x,y) = \left|\frac{\mathrm{d}Z}{\mathrm{d}z}\right|$$



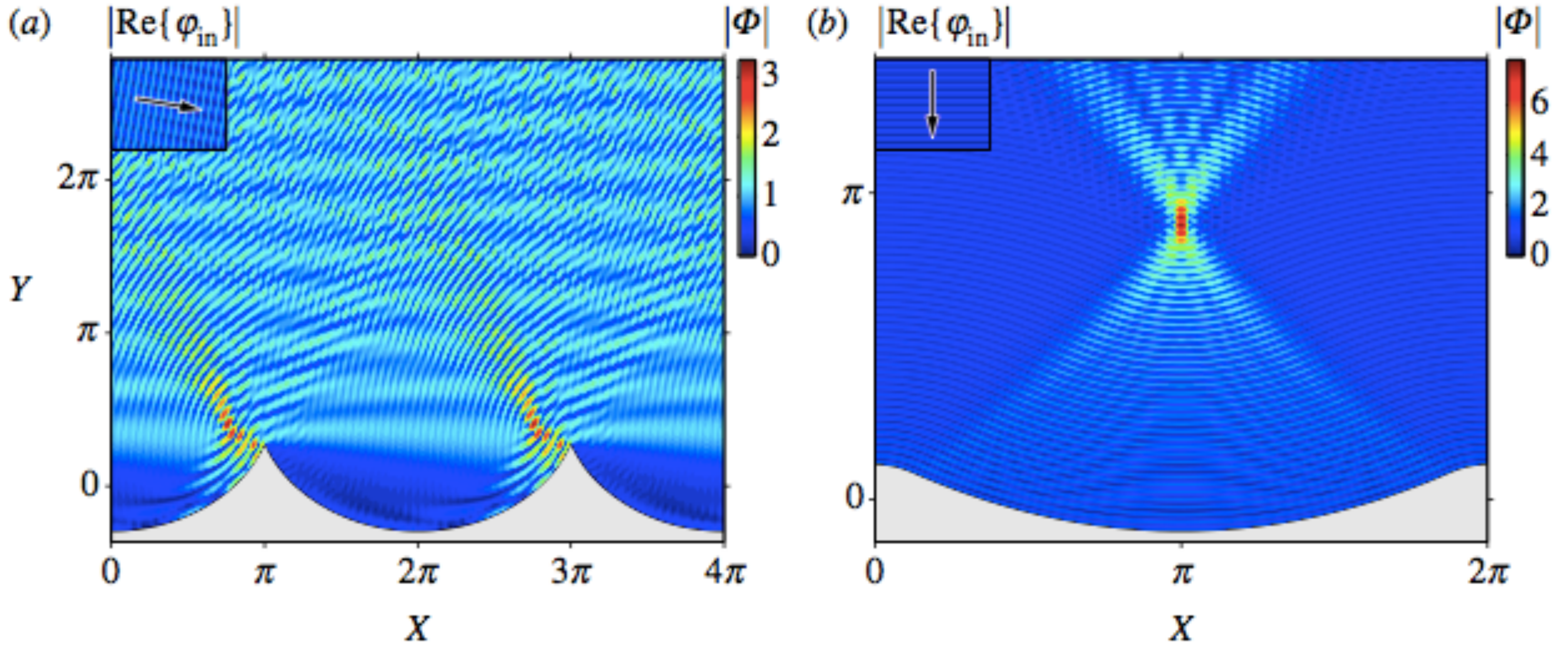
Riccati eq. for the Admittance matrix (DtN operator)

dY/dy = f (Y , y)

G. Favreau & V.P. , PRSA 2015

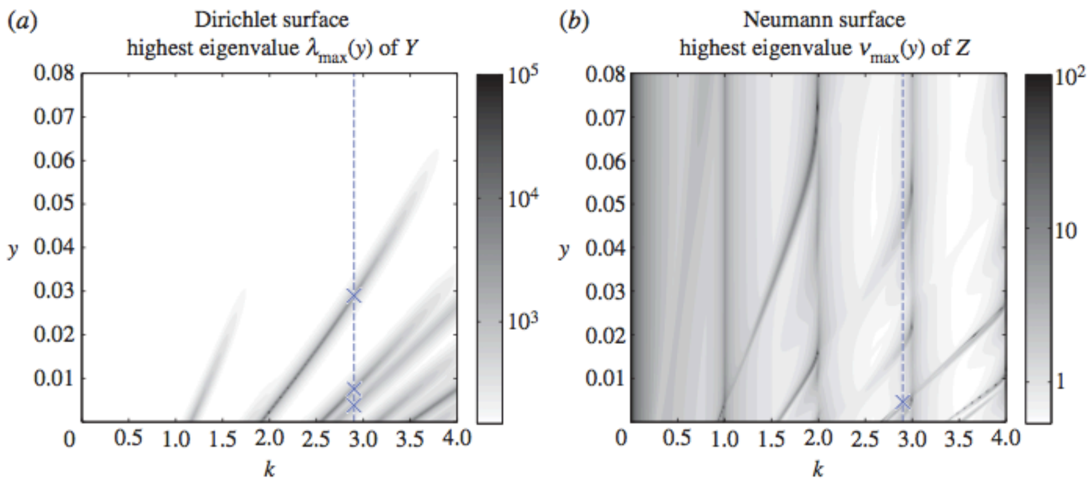
Remark (rough surface)

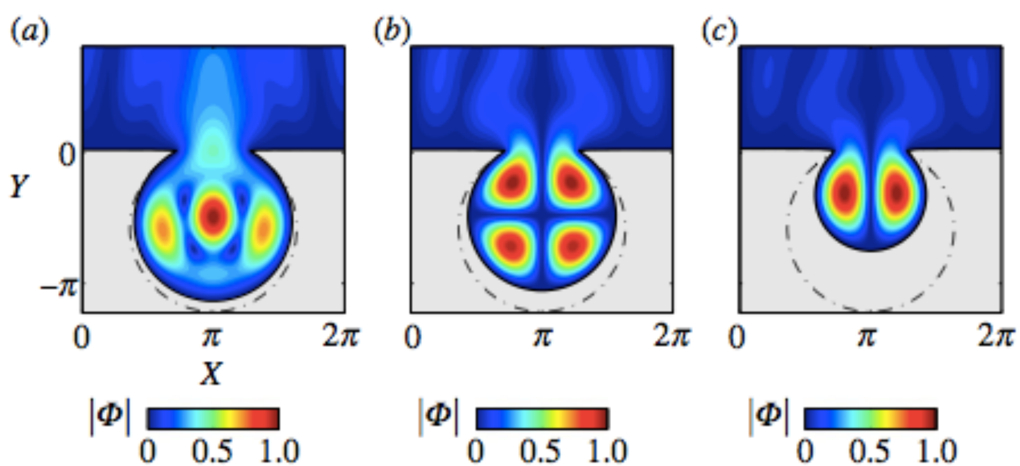
Multimodal Riccati to compute the scattering by the surface



Remark (rough surface)

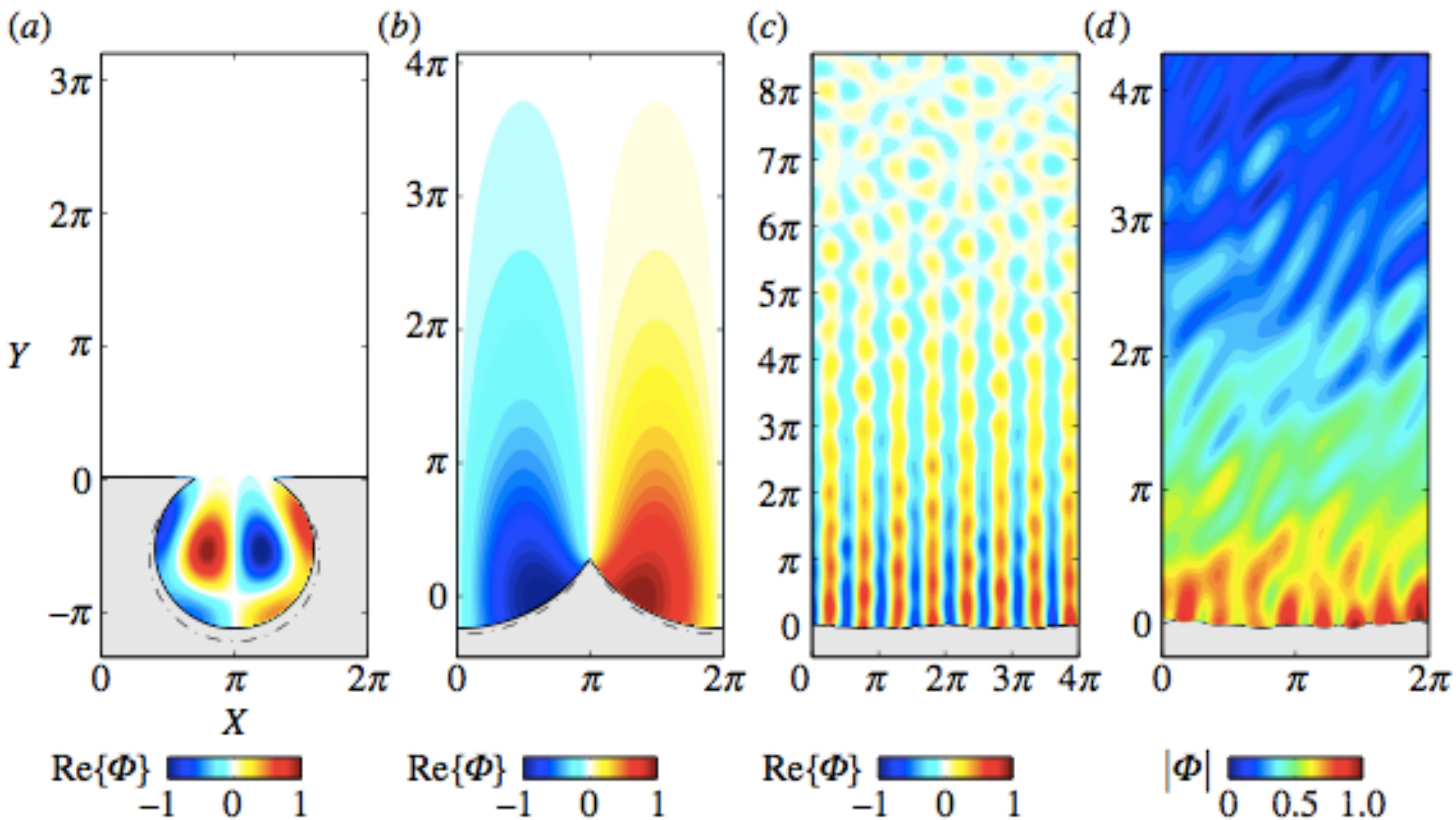
From Riccati : Eigenvalues of Y(y) can give total absorption or trapped surface waves





Remark (rough surface)

Exemples of surface waves “trapped by the roughness”
(also called BIC)



V. Pagneux, Multimodal admittance method and singularity behaviour at high frequencies, *J. Comp. App. Maths*. **234** (6), 1834-1841 (2010)