A ”milder” version of Calderon’s inverse problem for anisotropic conductivities and partial data

El Maati Ouhabaz, Univ. Bordeaux
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  One solves the Dirichlet problem:

$$\begin{cases} -\Delta u + Vu = 0 \ \text{ in } \Omega, u \in H^1(\Omega) \\ u = \varphi \ \text{ on } \partial\Omega \end{cases}$$

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- Some known (positive) results:
  - Sylvester-Uhlmann (87): $a, b \in C^2$ and $n \geq 3$,
  - Greenleaf-Lassas-Uhlmann ('03): $a, b \in C^{1+\epsilon}$,
  - Haberman-Tataru ('13): $a, b \in C^1$ (even Lipschitz in some cases),
  - Nachman ('96): $a, b \in C^2$ and $n = 2$,
  - Astala-Päivärinta ('06): $a, b \in L^\infty$ and $n = 2$,
  - Uhlmann, Novikov (+....): Schrödinger case with $V_1, V_2 \in L^\infty$,

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Other aspects: reconstruction of the potential or conductivity or the geometry of a manifold from the Dirichlet-to-Neumann operator on the boundary.

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  - Imanuvilov-Uhlmann-Yamamoto ('10): $n = 2$, V_1, V_2 smooth and $\Gamma_N = \partial\Omega \setminus \Gamma_D$.
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Calderon's inverse problem for partial data is in general open for $n \geq 3$ both for Schrödinger and conductivity cases.

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$$\begin{cases} L_a u = 0 \text{ in } \Omega, u \in H^1(\Omega) \\ u = \varphi \text{ on } \partial\Omega \end{cases}$$

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- Astala-Lassas-Päivärinta ('05): Uniqueness up to a diffeomorphism ($n = 2$ and L^∞ coefficients),
- Behrndt-Rohleder ('12): $a = (a_{kj}), b = (b_{kj})$ Lipschitz, $\Gamma \subset \partial\Omega$ and the Dirichlet-to-Neumann operators $\mathcal{N}_a(\lambda) = \mathcal{N}_b(\lambda)$ on Γ and λ in a set having an accumulation point ($\mathcal{N}_a(\lambda)$ is the Dirichlet-to-Neumann operator for $L_a - \lambda I$). Then L_a and L_b, subject to Dirichlet boundary conditions, are unitarily equivalent, i.e. $L_a = \mathcal{U} L_b \mathcal{U}^{-1}$, $\mathcal{U}$ is a unitary operator on $L^2(\Omega)$.

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Let $a = \{a_{kj}, a_0\}$ and $b = \{b_{kj}, b_0\}$ be bounded functions on Ω such that a_{kj} and b_{kj} satisfy the usual ellipticity condition. If $d \geq 3$ we assume in addition that the coefficients a_{kj}, b_{kj}, a_0, b_0 are Lipschitz continuous.

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Suppose that $\mathcal{N}_a(\lambda) = \mathcal{N}_b(\lambda)$ on Γ_1 for all λ in a set having an accumulation point in $\rho(L_a^D) \cap \rho(L_b^D)$. Then:

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$i)$ The operators L_a and L_b endowed with Robin boundary conditions are unitarily equivalent.

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i) The operators L_a and L_b endowed with Robin boundary conditions are unitarily equivalent.
ii) The operators L_a and L_b endowed with mixed boundary conditions (Dirichlet on Γ_0 and Neumann on Γ_1) are unitarily equivalent.

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$ii)$ The operators L_a and L_b endowed with mixed boundary conditions (Dirichlet on Γ_0 and Neumann on Γ_1) are unitarily equivalent.
$iii)$ The operators L_a and L_b endowed with Dirichlet boundary conditions are unitarily equivalent.
In addition, for Robin or mixed boundary conditions, the eigenfunctions associated to the same eigenvalue $\lambda \notin \sigma(L_a^D) = \sigma(L_b^D)$ coincide on the boundary of Ω.

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- We extend the result of [Berhndt-Rohleder] to deal with other boundary conditions (Robin and mixed ones).
- The meaning of the latest assertion is: for every $\lambda \in \sigma(L_a^\mu) = \sigma(L_b^\mu)$ with $\lambda \notin \sigma(L_a^D) = \sigma(L_b^D)$, the sets $\{\mathrm{Tr}(u), u \in \mathrm{Ker}(\lambda I - L_a^\mu)\}$ and $\{\mathrm{Tr}(v), v \in \mathrm{Ker}(\lambda I - L_b^\mu)\}$ coincide.

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$$\sum_{k,j=1}^{n} a_{kj}(x)\xi_k\xi_j \geq \delta|\xi|^2 \; a.e. x \in \Omega \, \forall \xi \in \mathbb{R}^n (\delta > 0).$$

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- Dirichlet boundary conditions (L_a^D): $u = 0$ on $\partial\Omega$.
- Mixed boundary conditions (L_a^M):

$$\begin{cases} \mathrm{Tr}(u) = 0 & \text{on } \Gamma_0 \\ \displaystyle\sum_{j=1}^{d}\sum_{k=1}^{n} a_{kj}\partial_k u.n_j = 0 & \text{on } \Gamma_1. \end{cases}$$

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- Robin boundary condition (L_a^μ):

$$\begin{cases} \mathrm{Tr}(u) = 0 & \text{on } \Gamma_0 \\ \displaystyle\sum_{j=1}^{d}\sum_{k=1}^{n} a_{kj}\partial_k u . n_j = \mu \, \mathrm{Tr}(u) & \text{on } \Gamma_1 . \end{cases}$$

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This allows to define on $L^2(\partial\Omega)$ the symmetric form:

$$\mathfrak{b}(\varphi, \psi) := \sum_{k,j=1}^{n} \int_\Omega a_{kj} \partial_j u \partial_k v dx + \int_\Omega a_0 uv dx,$$

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for $u, v \in V_H$ with $\varphi = \mathrm{Tr}(u)$ and $\psi = \mathrm{Tr}(v)$.
The operator associated with the form $\mathfrak{b}$ is given by:

$$\mathcal{N}_a \varphi = \psi \Leftrightarrow \forall h \in \mathrm{Tr}(V) : \mathfrak{b}(\varphi, h) = \int_{\partial\Omega} \psi h d\sigma \Leftrightarrow$$

$$\varphi = 0 \text{ on } \Gamma_0,\ \psi = \sum_{k,j} a_{kj} \partial_k u . n_j \ \text{ on } \Gamma_1 = \partial\Omega \setminus \Gamma_0.$$

This is the Dirichlet-to-Neumann operator with partial data (given on Γ_1).

Set

$$H := \overline{D(\mathfrak{b})}^{L^2(\partial\Omega)} = L^2(\Gamma_1) \oplus \{0\}.$$

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b) Suppose that $a_0 \geq 0$*. Then* $(e^{-t\mathcal{N}_a})_{t\geq 0}$ *is a sub-Markovian semigroup.*

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b) Suppose that $a_0 \geq 0$. Then $(e^{-t\mathcal{N}_a})_{t\geq 0}$ is a sub-Markovian semigroup.
c) Suppose that $a_0 \geq 0$. Then $(e^{-t\mathcal{N}_a})_{t\geq 0}$ is non-decreasing as a function of Γ_1.

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a) The semigroup $(e^{-t\mathcal{N}_a})_{t\geq 0}$ is positive (i.e., it maps non-negative functions of $L^2(\partial\Omega)$ into non-negative functions).
b) Suppose that $a_0 \geq 0$. Then $(e^{-t\mathcal{N}_a})_{t\geq 0}$ is a sub-Markovian semigroup.
c) Suppose that $a_0 \geq 0$. Then $(e^{-t\mathcal{N}_a})_{t\geq 0}$ is non-decreasing as a function of Γ_1. It is a non-increasing function of a_0.

The proof of a) and b) are based on the theory of Dirichlet forms.

Set

$$H := \overline{D(\mathfrak{b})}^{L^2(\partial\Omega)} = L^2(\Gamma_1) \oplus \{0\}.$$

Then $\mathcal{N}_a$ is a self-adjoint operator on H (not densely defined on $L^2(\partial\Omega)$).
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Assertion c) is based on criteria for domination of semigroups ([Ou' 95]).

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Proof:

$$S : \mathrm{Ker}(\lambda - L_a^\mu) \to \mathrm{Ker}(\mu - \mathcal{N}_a(\lambda)), \; u \mapsto \mathrm{Tr}(u)$$

is an isomorphism.

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Next we prove

Lemma

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$$\begin{aligned}
\lambda \int_{\Omega} u^{\mu+h}\overline{u^{\mu}}\, dx &= (L^{\mu+h}_{a} u^{\mu+h}, u^{\mu}) \\
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Take $h \to 0$.

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Finally, the unitary operator $\mathcal{U}$ s.t. $L^{\mu}_b = \mathcal{U} L^{\mu}_a \mathcal{U}^{-1}$ is constructed by:

$$\mathcal{U} : L^2(\Omega) \to L^2(\Omega), \; f_k \mapsto g_k$$

where (f_k) and (g_k) are the eigenfunctions of L^{μ}_a and L^{μ}_b (these are o.n. bases of $L^2(\Omega)$).

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As a consequence, $\lambda_{a,k}^\mu \to \lambda_{a,k}^D$ as $\mu \to -\infty$, where $\sigma(L_a^D) = (\lambda_{a,k}^D)_k$. The same holds for L_b^D and the similarity of L_a^D and L_b^D follows.