Spectral analysis of the magnetic Laplacian with vanishing magnetic field

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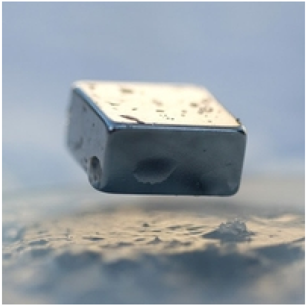
Superconductivity



- **The aim: spectral analysis of the elliptic operator** $\mathcal{P}_{h,\mathbf{A},\Omega} = (-ih\nabla + \mathbf{A})^2$

$$\mathcal{P}_{h,\mathbf{A},\Omega} = (-ih\nabla_x + \mathbf{A}(x))^2 = \sum_{j=1}^{d} (hD_{x_j} + A_j(x))^2, \quad \boxed{D_{x_j} = -i\partial_{x_j}}$$

  - Dimension: $d = 2$
  - $\Omega \subseteq \mathbb{R}^2$ open set
  - $\mathbf{A} = (A_1, A_2) \in \mathscr{C}^\infty(\overline{\Omega}, \mathbb{R}^2)$;
  - h: the semiclassical parameter
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- **Why is it interesting?**
  1. Physical applications in surface superconductivity (Ginzburg-Landau functional)
  2. Different from the electric case(?)
  3. Some connections with waveguides(?)

Existence of a bound state for an electric Laplacian...

The Lu-Pan operator: the self-adjoint Neumann realization on $\mathbb{R}^2_+$ of

$$\mathfrak{L}_\theta^{\mathsf{LP}} = -\Delta + V_\theta, \text{ with } V_\theta(s,t) = t\cos\theta - s\sin\theta \text{ and } \theta \in (0,\pi)$$

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For all $\theta \in (0,\pi)$ there exists an eigenvalue of $\mathfrak{L}_\theta^{\mathsf{LP}}$ below the essential spectrum wich equals $[1,+\infty)$.

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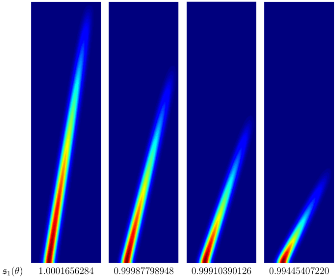
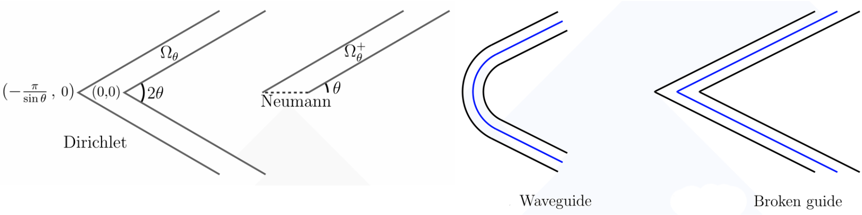


Figure : First eigenfunction of $\mathfrak{L}_\theta^{\mathsf{LP}}$ for $\theta \in \{\frac{k\pi}{2}, k \in \{0.9, 0.85, 0.8, 0.7\}\}$

... recall Duclos and Exner's result

Waveguide of width $\varepsilon > 0$ **in** $2D$**:** $\{\Phi(s,t) = \gamma(s) + t\mathbf{n}(s), (s,t) \in \mathbb{R} \times (-\varepsilon, \varepsilon)\}$



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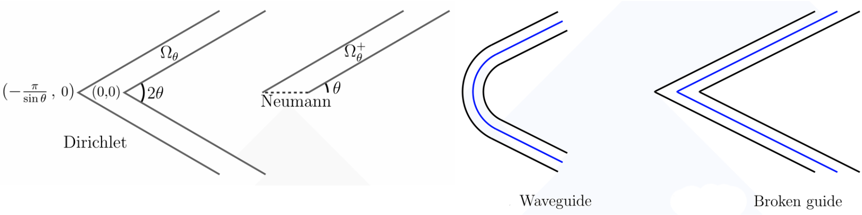

Known result:

For a waveguide straight at infinity but not everywhere, there is always an eigenvalue below the essential spectrum cross section (in the case of a circular cross section in dimensions two and three).

P. Duclos, P. Exner, *Curvature-induced bound states in quantum waveguides in two and three dimensions.* (1995).

Magnetic waveguides

Waveguide: a tube $\Omega_\varepsilon \subseteq \mathbb{R}^d$ about an unbounded curve γ

- d: dimension ≥ 2
- $\varepsilon > 0$ shrinking parameter
- $\varepsilon\omega$: the crosssection with $\omega \subset \mathbb{R}^{d-1}$ bounded and simply connected

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Spectral analysis of the magnetic operator $\mathfrak{L}^{[d]}_{\varepsilon,b\mathbf{A}}$ with Dirichlet boundary :

$$\mathfrak{L}^{[d]}_{\varepsilon,b\mathbf{A}} = (-i\nabla_{\mathrm{x}} + b\mathbf{A}(\mathrm{x}))^2 \text{ on } \mathrm{L}^2(\Omega_\varepsilon, \mathrm{dx}),\ \ b > 0$$

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What is the spectral influence of a magnetic field on a waveguide?

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$$\mathscr{C}^{\infty}(\overline{\Omega},\mathbb{C}) \ni u \mapsto \mathcal{Q}_{h,\mathbf{A},\Omega} = \int_{\Omega} |(-ih\nabla + \mathbf{A})u|^2 \, \mathrm{dx}$$

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- **Problematic: behaviour of the eigenvalues and the eigenfunctions when $h \to 0$**

$$(\lambda_n(h), \psi_{n,h}) \underset{h\to 0}{\sim} ?$$

Literature

S. Fournais, B. Helffer, *Spectral methods in Surface Superconductivity*, Progress in Nonlinear Differential Equations and their Applications, 77, Birkhäuser Boston Inc., Boston, MA, 2010.

N. Raymond, *Little Magnetic Book*. Preprint, 2016.

Non vanishing magnetic field:

Constant magnetic field $\mathbf{B} \equiv 1$:

- Bolley, Helffer (1997), Bauman-Phillips-Tang (1998), del Pino, Felmer, Sternberg (2000), (2D, disc),
- Helffer, Morame (2001), (2D, smooth boundary),
- Helffer, Morame (2004), (3D, smooth boundary),
- Bonnaillie (2005), (2D, corners),
- Fournais, Persson (2011), (3D, balls).

Non vanishing and variable magnetic field $\mathbf{B}$:

- Lu, Pan (1999) ; Raymond (2009) (2D, smooth boundary),
- Lu, Pan (2000) ; Raymond (2010) ; Helffer, Kordyukov (2013), (3D, smooth boundary),
- Bonnaillie-Noël (2005), Bonnaillie-Noël, Dauge (2006), Bonnaillie-Noël, Fournais (2007), (2D, corners),
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- Bonnaillie-Noël, Dauge, N. Popoff (2016), (3D, corners).

Vanishing magnetic field:

- Montgomery (1995), (the first case when the model of cancellation appears),
- Helffer, Morame (1996) (behaviour of the ground state in hypersurface),
- Pan, Kwek (2002), (2D, Neumann boundary condition),
- Helffer, Kordyukov (2009), (hypersurface),
- Dombrowski, Raymond (2013), (cancellation along a closed and smooth curve in the whole plane),
- Bonnaillie-Noël, Raymond (2015), (broken line of cancellation inside Ω, Neumann boundary condition),
- Attar, Helffer, Kachmar (2015), (minimizing of the energy when the Ginzburg-Landau parameter tends to infinity, Neumann boundary condition).

Why considering vanishing magnetic fields?

- **Mathematical reasons:** analyze the spectral influence of the cancellation of the magnetic field in the semiclassical limit.
- **Study of "magnetic waveguides":**

N. Dombrowski, F. Germinet, G. Raikov, *Quantization of edge currents along magnetic barriers and magnetic guides*. (2011).

... inspired by the physical considerations:

J. Reijniers, , A. Matulis, K. Chang, F. Peeters, *Confined magnetic guiding orbit states*. (2002).

M. Hara, A. Endo, S. Katsumoto, Y. Iye, *Transport in two-dimensional electron gas narrow channel with a magnetic field gradients*. (2004).

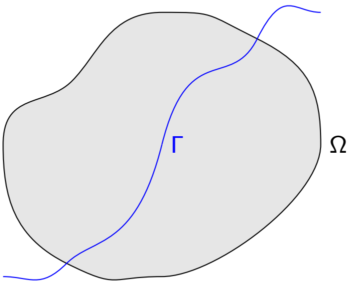


Figure : Domain Ω and the (smooth) vanishing curve Γ.

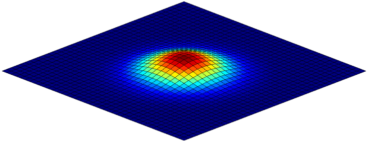
Assumptions:

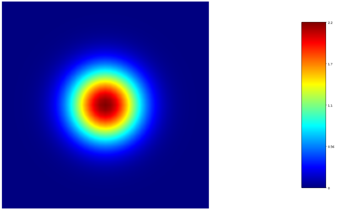
- $\sharp(\Gamma \cap \partial\Omega) < \infty$ and Γ is non tangent to $\partial\Omega$
- $|\nabla \mathbf{B}(\mathrm{x})| \neq 0, \ \forall\, \mathrm{x} \in \Gamma$

Localisation phenomena : concentration of the modes when $h \to 0$

Ground state g_1 of the electric Laplacian $-h^2\Delta + |x|^2$ in $\mathbb{R}^2$:

$$g_1(x) = \frac{1}{\sqrt{h}} \exp\left(-\frac{|x|^2}{2h}\right), h = \frac{1}{5}$$





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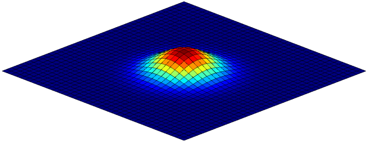

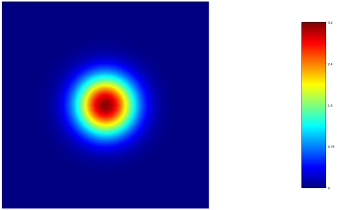

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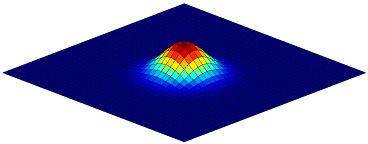

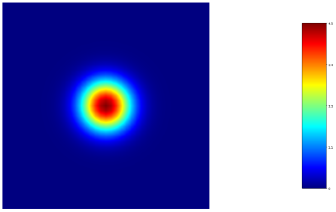

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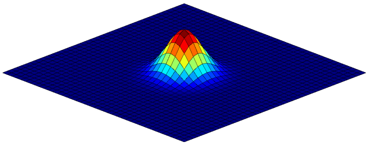

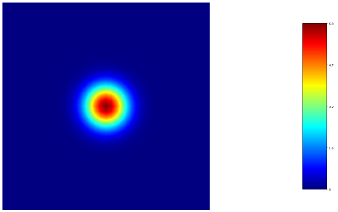

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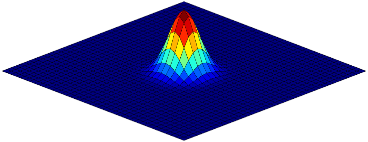
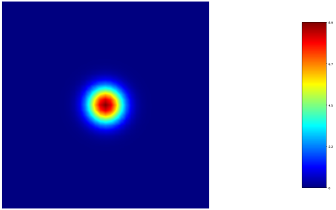
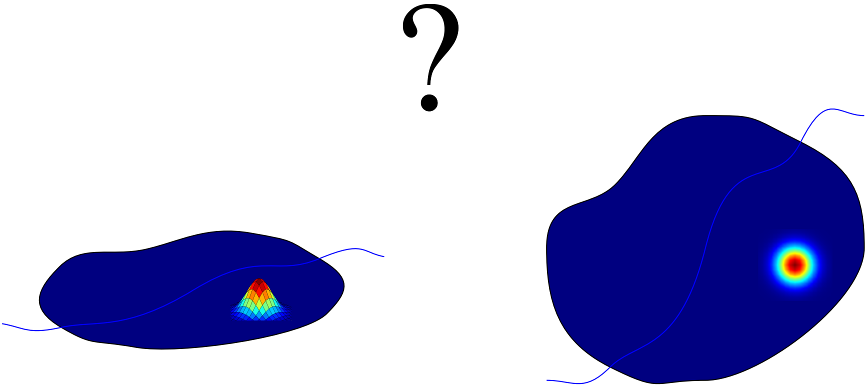




Where does the first eigenfunction(s) localize in the semiclassical limit?



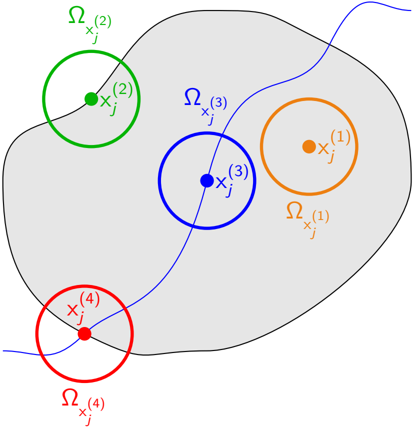
Different "areas" on Ω

(1) $\boldsymbol{\Omega} \backslash \boldsymbol{\Gamma}$

(2) $\partial\boldsymbol{\Omega} \backslash \boldsymbol{\Gamma}$

(3) $\boldsymbol{\Gamma} \backslash \partial\boldsymbol{\Omega}$

(4) $\partial\boldsymbol{\Omega} \cap \boldsymbol{\Gamma}$



"Zoom" on areas

$$" \mathcal{P}_{h,\mathbf{A},\Omega} \underset{h\to 0}{\sim} \mathcal{P}_{h,\mathbf{A},\Omega_{x_j^{(1)}}} \oplus \mathcal{P}_{h,\mathbf{A},\Omega_{x_j^{(2)}}} \oplus \mathcal{P}_{h,\mathbf{A},\Omega_{x_j^{(3)}}} \oplus \mathcal{P}_{h,\mathbf{A},\Omega_{x_j^{(4)}}} "$$

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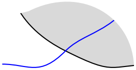

$x = \Phi(s, t)$

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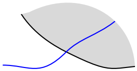

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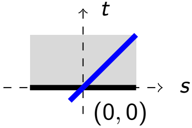
2. **Change of variable $\Rightarrow$ Flat geometry:**



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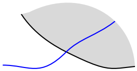

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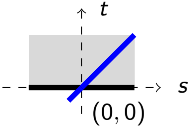
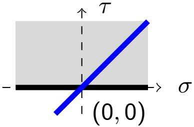


3. **Scaling:** $\mathrm{X} - \mathrm{x}_j = \frac{\mathrm{x}-\mathrm{x}_j}{h^\beta}$



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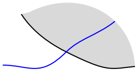
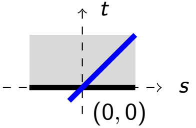
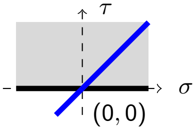


$$\mathrm{x} = \Phi(s,t)$$

2. **Change of variable $\Rightarrow$ Flat geometry:**



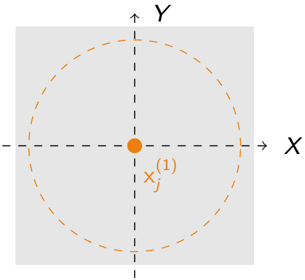
3. **Scaling:** $\mathrm{X} - \mathrm{x}_j = \frac{\mathrm{x}-\mathrm{x}_j}{h^\beta}$



4. **Approximation on the whole (half) space + Linearisation :**

$$\mathcal{P}_{h,\mathbf{A},\Omega_{x_j}} \underset{h\to 0}{\sim} h^p \mathcal{P}_{1,\mathbf{A}^{\mathrm{mod}},\mathbb{R}^2_{(+)}}$$

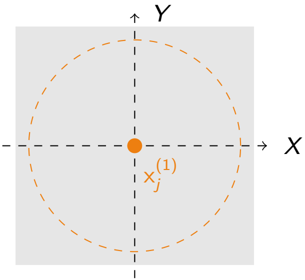
Model operator and operator of reference



The magnetic Laplacian $\mathcal{P}_{1,\mathbf{A},\mathbb{R}^2}$ ($h = 1$) in the model case when $\mathbf{B} \equiv 1$:

$$D_Y^2 + (D_X - Y)^2$$

Model operator and operator of reference



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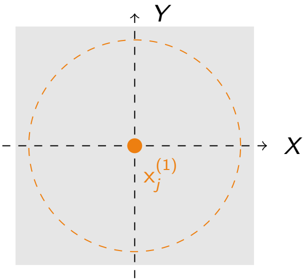
By unitary transforms we are reduced to the harmonic oscillator:

$$\mathcal{H} = D_Y^2 + Y^2, \text{ on } \mathbb{R}$$

Model operator and operator of reference



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By unitary transforms we are reduced to the harmonic oscillator:

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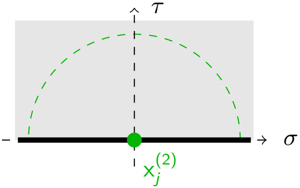
Bottom of the spectrum of the operator $\mathcal{H}$:

$$\inf \mathrm{Sp}(\mathcal{H}) = 1$$

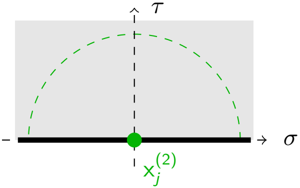
Model operator and operator of reference



The magnetic Laplacian $\mathcal{P}_{1,\mathbf{A},\mathbb{R}^2_+}$ ($h = 1$) in the model case when $\mathbf{B} \equiv 1$:

$$D_\tau^2 + (D_\sigma - \tau)^2$$

Model operator and operator of reference



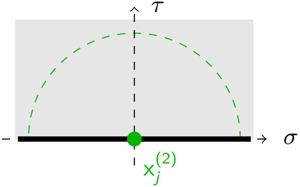
The magnetic Laplacian $\mathcal{P}_{1,\mathbf{A},\mathbb{R}^2_+}$ ($h=1$) in the model case when $\mathbf{B} \equiv 1$:

$$D_\tau^2 + (D_\sigma - \tau)^2$$

By unitary transforms we are reduced to the De Gennes operator:

$$\mathcal{G}(\xi) = D_\tau^2 + (\tau - \xi)^2 \text{ on } \mathbb{R}_+, \text{ Neuman boundary condition}$$

Model operator and operator of reference



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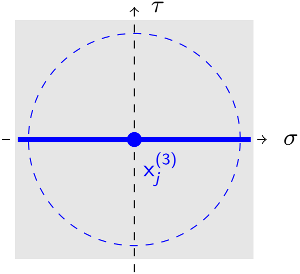
Bottom of the spectrum of the operator $\mathcal{G}(\xi)$:

$$\mu_1(\xi) = \inf \mathrm{Sp}(\mathcal{G}(\xi))$$

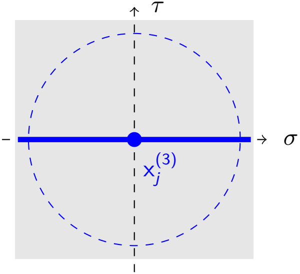
Model operator and operator of reference



The magnetic Laplacian $\mathcal{P}_{1,\mathbf{A},\mathbb{R}^2}$ ($h=1$) in the model case when $\mathbf{B}(\sigma,\tau)=\tau$:

$$D_\tau^2 + \left(D_\sigma - \frac{\tau^2}{2}\right)^2$$

Model operator and operator of reference



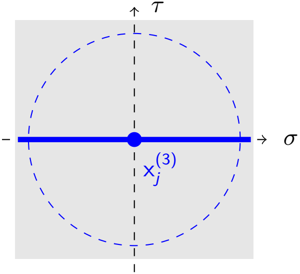
The magnetic Laplacian $\mathcal{P}_{1,\mathbf{A},\mathbb{R}^2}$ ($h = 1$) in the model case when $\mathbf{B}(\sigma, \tau) = \tau$:

$$D_\tau^2 + \left(D_\sigma - \frac{\tau^2}{2}\right)^2$$

By unitary transforms we are reduced to the Montgomery operator:

$$\mathcal{M}(\eta) = D_\tau^2 + \left(\frac{\tau^2}{2} - \eta\right)^2 \text{ on } \mathbb{R}$$

Model operator and operator of reference



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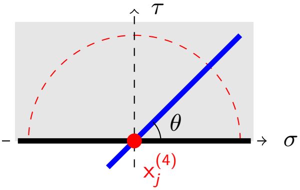
Bottom of the spectrum of the operator $\mathcal{M}(\eta)$:

$$\nu_1(\eta) = \inf \mathrm{Sp}(\mathcal{M}(\eta))$$

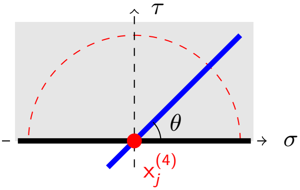
Model operator and operator of reference



The magnetic Laplacian $\mathcal{P}_{1,\mathbf{A},\mathbb{R}^2_+}$ ($h=1$) in the model case when:

$$\mathbf{B}(\sigma,\tau)=\tau\cos\theta-\sigma\sin\theta.$$

Model operator and operator of reference



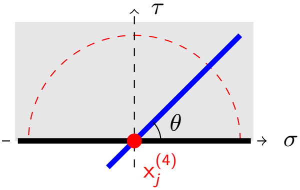
The magnetic Laplacian $\mathcal{P}_{1,\mathbf{A},\mathbb{R}^2_+}$ ($h = 1$) in the model case when:

$$\mathbf{B}(\sigma,\tau) = \tau\cos\theta - \sigma\sin\theta.$$

We get the Pan and Kwek operator:

$$\mathcal{K}_\theta = D_\tau^2 + \left(D_\sigma + \sigma\tau\sin\theta - \frac{\tau^2}{2}\cos\theta\right)^2 \text{ on } \mathbb{R}^2_+, \text{ Neumann boundary condition}$$

Model operator and operator of reference



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Bottom of the spectrum of the operator $\mathcal{K}_\theta$:

$$\inf \mathrm{Sp}(\mathcal{K}_\theta) = \zeta_1^\theta$$

Properties of the Pan and Kwek operator

Proposition:

$$\inf \mathrm{Sp}_{\mathrm{ess}}(\mathcal{K}_\theta) = \mathrm{M}_0 = \inf \mathrm{Sp}_{\mathrm{ess}} \mathcal{P}_{1,\mathbf{A},\mathbb{R}^2}$$

Proposition:

- $\zeta_1^0 = \zeta_1^\pi = \mathrm{M}_0$
- $\zeta_1^\theta < \mathrm{M}_0$, for all $\theta \in (0, \pi)$

X.-B. Pan, K.-H. Kwek, *Schrödinger operators with non-degenerately vanishing magnetic fields in bounded domains.* (2002).

Proposition:

For all $\theta \in (0, \pi)$, ζ_1^θ is a eigenvalue and the associated eigenfunctions belong to $\mathscr{S}(\overline{\mathbb{R}^2_+})$.

Case	Operator of reference	Infimum of the spectrum
(1)	$\mathcal{H} = D_Y^2 + Y^2$ on $\mathbb{R}$	1
(2)	$\mathcal{G}(\xi) = D_\tau^2 + (\tau - \xi)^2$ on $\mathbb{R}_+$ with Neumann boundary condition	$\inf_{\xi \in \mathbb{R}} (\mu_1(\xi)) = \Theta_0$
(3)	$\mathcal{M}(\eta) = D_\tau^2 + \left(\frac{\tau^2}{2} - \eta\right)^2$ on $\mathbb{R}$	$\inf_{\eta \in \mathbb{R}} (\nu_1(\eta)) = \mathrm{M}_0$
(4)	$\mathcal{K}_\theta = D_\tau^2 + \left(D_\sigma + \sigma\tau \sin\theta - \frac{\tau^2}{2}\cos\theta\right)^2$ on $\mathbb{R}_+^2$ with Neumann boundary condition	ζ_1^θ

Numerical computations:

- $\Theta_0 = \mu_1(\xi_0) \approx 0.5901$, with $\xi_0 = \sqrt{\Theta_0} \approx 0.7682$
- $\mathrm{M}_0 = \nu_1(\eta_0) \approx 0.5698$, with $\eta_0 \approx 0.35$
- $\zeta_1^{\frac{\pi}{2}} \approx 0.5494$

Back to the operator $\mathcal{P}_{h,\mathbf{A},\Omega}$: summary of the operator hierarchy

$$\mathrm{x}_j^{(\ell)} \in \Omega\backslash\Gamma, \partial\Omega\backslash\Gamma, \Gamma\backslash\partial\Omega, \partial\Omega\cap\Gamma$$

Case (ℓ)	Operator h dependant	Scaling β, h^p, $\mathbb{R}^2_{(+)}$	Infimum of the spectrum
(1)	$h^2 D_y^2 + (hD_y - \lvert\mathbf{B}(\mathrm{x}_j^{(1)})\rvert y)^2$ on $\mathbb{R}^2$	$\frac{1}{2}$, h, $\mathbb{R}^2$	$1\lvert\mathbf{B}(\mathrm{x}_j^{(1)})\rvert h$
(2)	$h^2 D_t^2 + (hD_s - \lvert\mathbf{B}(\mathrm{x}_j^{(2)})\rvert t)^2$ on $\mathbb{R}^2_+$ with Neumann boundary condition	$\frac{1}{2}$, h, $\mathbb{R}^2_+$	$\Theta_0\lvert\mathbf{B}(\mathrm{x}_j^{(2)})\rvert h$
(3)	$h^2 D_t^2 + \left(hD_s - \lvert\nabla\mathbf{B}(\mathrm{x}_j^{(3)})\rvert \frac{t^2}{2}\right)^2$ on $\mathbb{R}^2$	$\frac{1}{3}$, $h^{4/3}$, $\mathbb{R}^2$	$\mathrm{M}_0\lvert\nabla\mathbf{B}(\mathrm{x}_j^{(3)})\rvert^{\frac{2}{3}} h^{\frac{4}{3}}$
(4)	$h^2 D_t^2 + \left(hD_s + \lvert\nabla\mathbf{B}(\mathrm{x}_j^{(4)})\rvert \left(st\sin\theta(\mathrm{x}_j^{(4)}) - \frac{t^2}{2}\cos\theta(\mathrm{x}_j^{(4)})\right)\right)^2$ on $\mathbb{R}^2_+$ with Neumann boundary condition	$\frac{1}{3}$, $h^{4/3}$, $\mathbb{R}^2_+$	$\zeta_1^{\theta(\mathrm{x}_j^{(4)})}\lvert\nabla\mathbf{B}(\mathrm{x}_j^{(4)})\rvert^{\frac{2}{3}} h^{\frac{4}{3}}$

Approximation of the bottom of the spectrum of $\mathcal{P}_{h,\mathbf{A},\Omega}$ when $h \to 0$

Theorem:

Under the condition

$$\inf_{x\in\partial\Omega\cap\Gamma} \zeta_1^{\theta(x)} |\nabla \mathbf{B}(x)|^{2/3} < \mathrm{M}_0 \inf_{x\in\Omega\cap\Gamma} |\nabla \mathbf{B}(x)|^{2/3}$$

we have two results:

1. **Equivalent of the first eigenvalue**

$$\lambda_1(h) = h^{4/3} \inf_{x\in\partial\Omega\cap\Gamma} \zeta_1^{\theta(x)} |\nabla \mathbf{B}(x)|^{2/3} + \mathcal{O}(h^{5/3}).$$

2. **Exponential concentration of the first eigenvector**

   There exist $C > 0$, $\alpha > 0$, $h_0 > 0$, s. t. for all $h \in (0, h_0)$

$$\int_\Omega e^{2\alpha h^{-1/3} d(x,\partial\Omega\cap\Gamma)} |\psi_{1,h}(x)|^2 \, \mathrm{d}x \leq C \|\psi_{1,h}\|^2_{L^2(\Omega)}.$$

Computation of the first ten eigenvalues for decreasing values of h

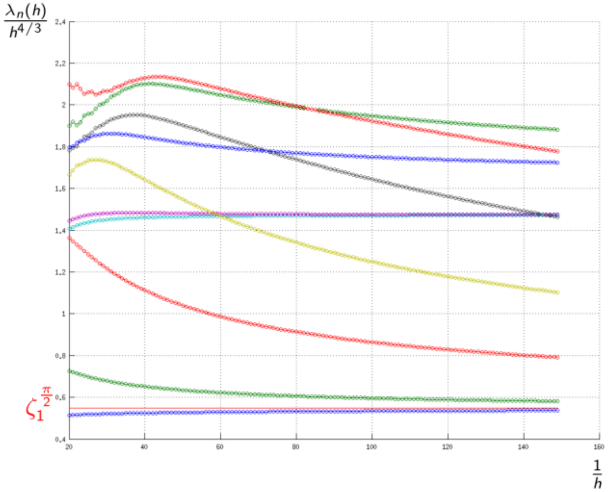
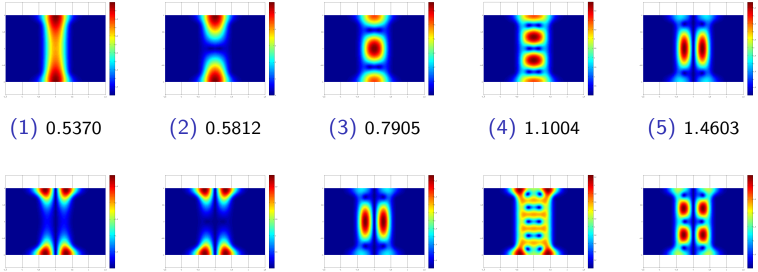


Figure : First ten eigenvalues $\lambda_n(h)$ rescaled by $h^{-4/3}$ according to $\frac{1}{h} \in [20, 150]$, $\mathbf{B}(s,t) = s$, $\Omega = [-\frac{3}{2}, \frac{3}{2}] \times [-1, 1]$. Finite elements, 24×16 quadrangular elements, $\mathbb{Q}_{10}$.

First ten eigenmodes in modulus, $h = \frac{1}{150}$

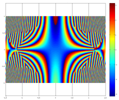


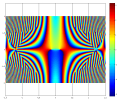
(1) 0.5370 (2) 0.5812 (3) 0.7905 (4) 1.1004 (5) 1.4603

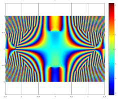
(6) 1.4728 (7) 1.4750 (8) 1.7237 (9) 1.7738 (10) 1.8811

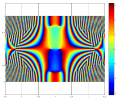
Figure : Modulus of $\psi_{n,h}$ and numerical value of $\lambda_n(h)h^{-4/3}$. Finite elements, 24×16 quadrangular elements, $\mathbb{Q}_{10}$.

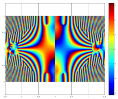
Phase of the first ten eigenmodes, $h = \frac{1}{150}$: high oscillations in $\frac{1}{h}$

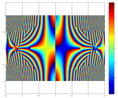

(1) 0.5370

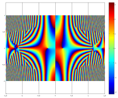

(2) 0.5812

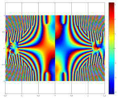

(3) 0.7905

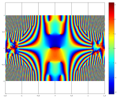

(4) 1.1004


(5) 1.4603


(6) 1.4728


(7) 1.4750


(8) 1.7237


(9) 1.7738

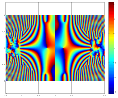

(10) 1.8811

Figure : Argument of $\psi_{n,h}$ and numerical value of $\lambda_n(h)h^{-4/3}$. Finite elements, 24×16 quadrangular elements, degree $\mathbb{Q}_{10}$.

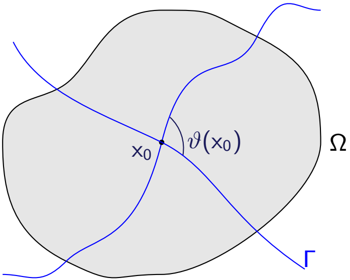
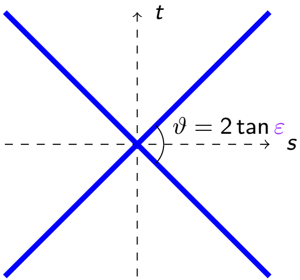


Figure : Domain Ω and the (smooth) vanishing curve Γ.

Assumptions:

- $\sharp(\Gamma \cap \partial\Omega) < \infty$ and Γ is non tangent to $\partial\Omega$
- $\exists ! x_0 \in \Gamma \backslash \partial\Omega$ with $|\nabla \mathbf{B}(x_0)| = 0$
- $|\nabla \mathbf{B}(x)| \neq 0, \ \forall \ x \in \Gamma\backslash\{x_0\}$
- $\mathrm{Hess}_{x_0}\mathbf{B} \neq 0$ and $\vartheta(x_0) \in (0, \pi)$



The magnetic Laplacian $\mathcal{P}_{1,\mathbf{A},\mathbb{R}^2}$ ($h = 1$) in the model case when $\mathbf{B}(\sigma, \tau) = t^2 - \varepsilon^2 s^2$:

$$\mathcal{X}_\varepsilon = D_\tau^2 + \left(D_\sigma + \varepsilon^2 \sigma^2 \tau - \frac{\tau^3}{3} \right)^2$$

Spectrum of $\mathcal{X}_\varepsilon$: $(\varkappa_n(\varepsilon))_{n \in \mathbb{N}^*} = \{\varkappa_1(\varepsilon) \leq \varkappa_2(\varepsilon) \leq \cdots\} \subseteq \mathbb{R}^+$, discrete

Approximation of the bottom of the spectrum of $\mathcal{P}_{h,\mathbf{A},\Omega}$ when $h \to 0$

Theorem:

We have:

1. **Equivalent of the first eigenvalue**

$$\lambda_1(h) = \mathrm{C}_0^{\mathbf{B}} h^{3/2} + \mathcal{O}(h^{7/4}),$$

where $\mathrm{C}_0^{\mathbf{B}} = \Xi(\mathrm{x}_0)^{1/2}\varkappa(\varepsilon(\mathrm{x}_0))$ with $\Xi(\mathrm{x}) = \frac{\sqrt{\mathrm{tr}^t \mathrm{Hess}\mathbf{B}(\mathrm{x})\mathrm{Hess}\mathbf{B}(\mathrm{x})}}{2\sqrt{\varepsilon(\mathrm{x})^4+1}}$ and $\varepsilon(\mathrm{x}_0)$ given by $\vartheta(\mathrm{x}_0) = 2\tan\varepsilon(\mathrm{x}_0)$.

2. **Exponential concentration of the first eigenvector**

There exist $C > 0$, $\alpha > 0$ and $h_0 > 0$, s. t. for all $h \in (0, h_0)$,

$$\int_\Omega e^{2\alpha h^{-1/4} d(\mathrm{x},\mathrm{x}_0)} |\psi_{1,h}(\mathrm{x})|^2 \,\mathrm{dx} \le C\|\psi_{1,h}\|^2_{\mathrm{L}^2(\Omega)}.$$

Numerical simulations: bottom of the spectrum of the symbol $\mathsf{X}_{\alpha,\xi}$ of $\mathcal{X}_\varepsilon$

$$\mathsf{X}_{\alpha,\xi} = D_\tau^2 + \left(\xi + \alpha^2\tau - \frac{\tau^3}{3}\right)^2, \text{ in } \mathbb{R}$$

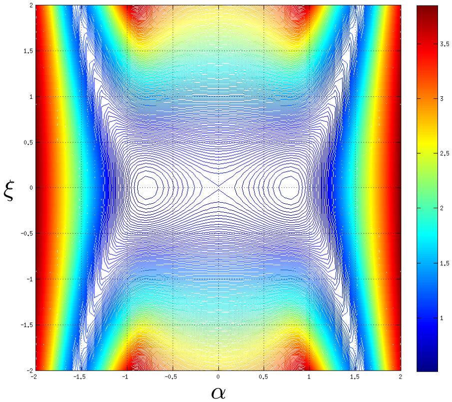


Figure : The "band function" $\varrho_1(\alpha,\xi) = \inf_{(\alpha,\xi)\in\mathbb{R}^2} \mathrm{Sp}\left(\mathsf{X}_{\alpha,\xi}\right)$.

Numerical simulations: first mode

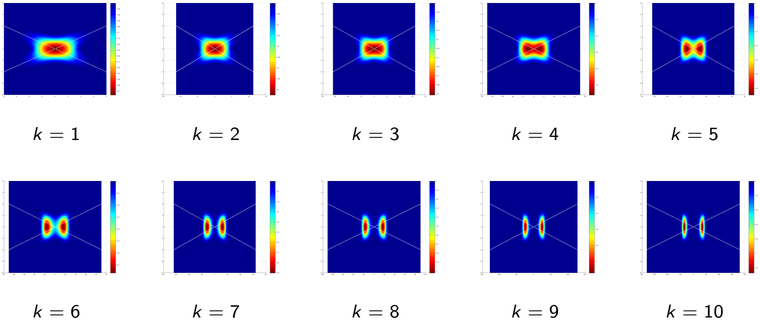


Figure : Modulus of the first mode $\psi_{n,h}$, for $\varepsilon = \left(\frac{1}{\sqrt{2}}\right)^k$. Finite elements, 48×6 quadrangular elements, degree $\mathbb{Q}_{10}$.

Approximation of the bottom of the spectrum of $\mathcal{X}_\varepsilon$ when $\varepsilon \to 0$

Theorem:

1. **Existence of the minimum for the operator symbol**
   There exist (α_0, ξ_0) in a compact set of $\mathbb{R}^2$ s. t.
   $$\varrho_1(\alpha_0, \xi_0) = \min_{(\alpha,\xi)\in\mathbb{R}^2} \varrho_1(\alpha, \xi).$$

2. **Equivalent for the bottom of the spectrum**
   For all $n \geq 1$ such that $\varkappa_n(\varepsilon) = \mathcal{O}(\varepsilon^0)$, there exist $C > 0$ and $h_0 > 0$ s. t. for all $h \in (0, h_0)$
   $$|\,\varkappa_n(\varepsilon) - \varrho_1(\alpha_0, \xi_0)| \leq C\varepsilon.$$

(Broken) Montgomery operator... recall (again) Duclos and Exner's result

- **Straight line of cancellation on the whole plane:**
  $D_\tau^2 + \left(D_\sigma + \sigma\tau\sin\theta - \frac{\tau^2}{2}\cos\theta\right)^2$ has essential spectrum.

(Broken) Montgomery operator... recall (again) Duclos and Exner's result

- **Straight line of cancellation on the whole plane:**
  $D_\tau^2 + \left(D_\sigma + \sigma\tau \sin\theta - \frac{\tau^2}{2}\cos\theta\right)^2$ has essential spectrum.
- **Straight line of cancellation on the half-plane with Neumann boundary condition:**
  $D_\tau^2 + \left(D_\sigma + \sigma\tau \sin\theta - \frac{\tau^2}{2}\cos\theta\right)^2$ has at least one eigenvalue $\forall\theta \in (0, \pi)$.

(Broken) Montgomery operator... recall (again) Duclos and Exner's result

- **Straight line of cancellation on the whole plane:**
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  $D_\tau^2 + \left(D_\sigma + \sigma\tau\sin\theta - \frac{\tau^2}{2}\cos\theta\right)^2$ has at least one eigenvalue $\forall\theta \in (0,\pi)$.
- **Broken line of cancellation on the whole plane:**
  $D_\tau^2 + \left(D_\sigma + \sigma\tau\sin\theta + \operatorname{sgn}(t)\frac{\tau^2}{2}\cos\theta\right)^2$.

(Broken) Montgomery operator... recall (again) Duclos and Exner's result

- **Straight line of cancellation on the whole plane:**
  $D_\tau^2 + \left(D_\sigma + \sigma\tau \sin\theta - \frac{\tau^2}{2}\cos\theta\right)^2$ has essential spectrum.
- **Straight line of cancellation on the half-plane with Neumann boundary condition:**
  $D_\tau^2 + \left(D_\sigma + \sigma\tau \sin\theta - \frac{\tau^2}{2}\cos\theta\right)^2$ has at least one eigenvalue $\forall\theta \in (0,\pi)$.
- **Broken line of cancellation on the whole plane:**
  $D_\tau^2 + \left(D_\sigma + \sigma\tau \sin\theta + \mathrm{sgn}(t)\frac{\tau^2}{2}\cos\theta\right)^2.$

Numerical conjecture:

There exists $\theta_0 \in (\frac{\pi}{4}, \frac{\pi}{2})$ s. t. the firt Rayleigh quotient is equal to the infimum of the essential spectrum (M_0) for all $\theta \in [\theta_0, \frac{\pi}{2})$ and strictly less for all $\theta \in (0, \theta_0)$.

V. Bonnaillie-Noel, N. Raymond, *Breaking a magnetic zero locus: model operators and numerical approach*. (2015).

Limit $\varepsilon \to 0$ for the Dirichlet Laplacian on the tube Ω_ε

- **The Dirichlet Laplacian in dimension** 2:

Known result:

The Dirichlet Laplacian on the tube Ω_ε converges (in a suitable sense) to:

$$\mathcal{L}^{\mathrm{eff}} = -\partial_s^2 - \frac{\kappa(s)}{4} \text{ on } \mathrm{L}^2(\gamma, \mathrm{d}s), \quad (\kappa \text{ is the curvature}).$$

P. Duclos, P. Exner, *Curvature-induced bound states in quantum waveguides in two and three dimensions*. (1995).

- **The Dirichlet Laplacian in twisted waveguide in dimension** 3:

Known result:

The Dirichlet Laplacian on the tube Ω_ε converges (in a suitable sense) to:

$$\mathcal{L}^{\mathrm{eff}} = -\partial_s^2 - \frac{\kappa(s)}{4} + C(\omega)\theta'(s)^2 \text{ on } \mathrm{L}^2(\gamma, \mathrm{d}s),$$

where θ is the angle function and $C(\omega)$ is a positive constant whenever ω is not a disk or annulus.

G. Bouchitte, M. L. Mascarenhas, L. Trabucho, *On the curvature and torsion effects inone dimensional waveguides*. (2007).

Limit $\varepsilon \to 0$ for $\mathfrak{L}^{[2]}_{\varepsilon,b\mathbf{A}} = (-i\nabla_x + b\mathbf{A}(x))^2$ on the tube Ω_ε, with $b \sim \varepsilon^{-1}$

$$\mathfrak{L}^{[2]}_{\varepsilon,b\mathbf{A}} \text{ on } \mathrm{L}^2(\mathbb{R}\times(-\varepsilon,\varepsilon),\mathrm{m}(s,t)\mathrm{d}s\mathrm{d}t) \sim \mathcal{L}^{[2]}_{\varepsilon,b\mathbf{A}} \text{ on } \mathrm{L}^2(\mathbb{R}\times(-\varepsilon,\varepsilon),\mathrm{d}s\mathrm{d}t)$$
$$\sim \mathcal{L}^{[2]}_{\varepsilon,b\mathbf{A}_\varepsilon} \text{ on } \mathrm{L}^2(\mathbb{R}\times(-1,1),\mathrm{d}s\mathrm{d}\tau)$$

Known result:

There exist K, ε_0, $C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$,

$$\left\| \left(\mathcal{L}^{[2]}_{\varepsilon,\varepsilon^{-1}\mathbf{A}_\varepsilon} - \varepsilon^{-2}\lambda_1^{\mathrm{Dir}}(\omega) + K \right)^{-1} - \left(\mathcal{L}^{\mathrm{eff},[2]}_\varepsilon - \varepsilon^{-2}\lambda_1^{\mathrm{Dir}}(\omega) + K \right)^{-1} \right\| \leq C\varepsilon,$$

where $\lambda_n(\omega)^{\mathrm{Dir}}$ is the n-th eigenvalue of the Dirichlet Laplacian $\Delta_\omega^{\mathrm{Dir}}$ on $\mathrm{L}^2(\omega)$, and

$$\mathcal{L}^{\mathrm{eff},[2]}_\varepsilon = -\varepsilon^{-2}\Delta_\omega^{\mathrm{Dir}} + \mathcal{T}^{[2]},$$

$$\mathcal{T}^{[2]} = -\partial_s^2 - \frac{\kappa(s)}{4} + \left(\frac{1}{3} + \frac{2}{\pi^2}\right)\mathbf{B}(\gamma(s))^2.$$

D. Krejčiřík, N. Raymond, *Magnetic effects in curved quantum waveguides.* (2013).

Counting of eigenvalues?

Known result:

For all waveguide with corner, there is a finite number of eigenvalues below the threshold of the essential spectrum.

M. Dauge, Y. Lafranche, N. Raymond, *Quantum waveguides with corners.* (2012).

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Number of eigenvalue(s) of $\mathcal{K}_\theta$ when $\theta \to 0$?

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M. Dauge, Y. Lafranche, N. Raymond, *Quantum waveguides with corners.* (2012).

Number of eigenvalue(s) of $\mathcal{K}_\theta$ when $\theta \to 0$? Numerical answer:

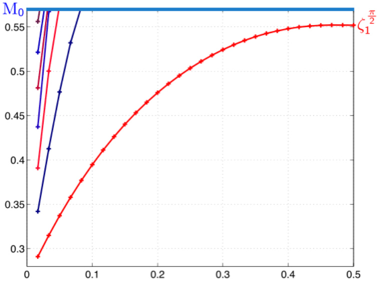

Figure : Eigenvalues ζ_n^θ below the bottom of the essential spectrum, for $\theta \in \{\frac{k\pi}{60}, 1 \leq k \leq 30\}$

V. Bonnaillie-Noel, N. Raymond, *Breaking a magnetic zero locus: model operators and numerical approach.* (2015).

Thank you!