On Lieb-Thirring inequalities, zeros of holomorphic functions, and waveguides

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Conference on Waveguides

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Plan of the talk

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4. Some open problems.

Few references

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- **2013 - ...** – C. Dubuisson, D. Sambou, ...
- **2014 - ...** – R. Frank, J. Sabin, B. Simon, J.C. Cuenin, ...

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A general scheme

Let $A_0 : H \to H$ be an operator on a Hilbert space, and $K : H \to H$ be an operator lying in $\mathcal{S}_p,\ 1 \le p < \infty$. Recall that

$$\mathcal{S}_p = \{K \in \mathcal{S}_\infty : ||K||_p^p := ||K||_{\mathcal{S}_p}^p = \sum_k s_k(A)^p < \infty\},$$

where $s_k(K) = \lambda_k(K^*K)^{1/2}$.

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Problem

What can one say about $\sigma_d(A)$ and its distributional characteristics ?

A general scheme

One needs to look at the so-called regularised determinant. For $K \in \mathcal{S}_p$ and $p \in \mathbb{N}^*$, define

$$\det{}_p(I+K) = \prod_k (1+\lambda_k) \exp\left(\sum_{j=1}^{p-1} \frac{(-1)^j}{j} \lambda_k^j\right),$$

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where $\lambda_k = \lambda_k(K)$.

Furthermore, consider the regularised perturbation determinant, *i.e.,*

$$G(\lambda) = \det{}_p(A-\lambda I)(A_0-\lambda I)^{-1} = \det{}_p(I + K(A_0-\lambda I)^{-1}).$$

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- $Z(G)$, the zero set of G, coincides with $\sigma_d(A)$ up to mutiplicities,
- there is a special bound of G on $\varrho(A_0)$, *i.e.*,

$$\begin{aligned} \log|G(\lambda)| &\leq \Gamma_p ||K(A_0-\lambda)^{-1}||_{\mathcal{S}_p}^p \leq \Gamma_p ||K||_{\mathcal{S}_p}^p ||(A_0-\lambda)^{-1}||^p \\ &\leq \Gamma_p \frac{||K||_{\mathcal{S}_p}^p}{d(\lambda, \sigma(A_0))^p} \end{aligned}$$

with $\lambda \in \varrho(A_0)$.

A general scheme

Now, let $\varphi : \mathbb{D} = \{z : |z| < 1\} \to \varrho(A_0)$ and $\psi : \varrho(A_0) \to \mathbb{D}$ be the conformal maps of the corresponding domains, $\psi = \varphi^{-1}$. Make a "change of variables" $\lambda = \varphi(z),\ z \in \mathbb{D}$.

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One comes to $g(z) = G(\varphi(z)) \in Hol(\mathbb{D})$ such that

$$\log|g(z)| \le \frac{K}{d^p(z, \mathbb{T})} \frac{d^r(z, E)}{d^q(z, F)}, \quad z \in \mathbb{D},\ p, q, r \ge 0,$$

and $E, F \subset \mathbb{T},\ \mathbb{T} = \{z : |z| = 1\},\ \#E, \#F < \infty$ and $E \cap F = \emptyset$. Of course,

$$d(z, E) = \inf_{t \in E} |z - t|,$$

so, for instance, $d(z, \mathbb{T}) = (1 - |z|),\ z \in \mathbb{D}$.

Zeros of holomorphic functions from different classes

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- Let $F \subset \mathbb{T}$, $\#F < \infty$.

Theorem (Borichev-Golinskii-K' 2009)

Let $f \in Hol(\mathbb{D})$, $|f(0)| = 1$, satisfy the growth condition

$$\log|f(z)| \le \frac{K}{(1-|z|)^p\, d^q(z,F)}$$

for $z \in \mathbb{D}$ and $p, q \ge 0$. Then for each $\tau > 0$ there is a positive constant $C_1 = C_1(p,q,F;\tau)$ such that

$$\sum_{\zeta \in Z(f)} (1-|\zeta|)^{p+1+\tau}\, d^{(q-1+\tau)_+}(\zeta, F) \le C_1 \cdot K.$$

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$$\sum_{\zeta \in Z(f)} (1-|\zeta|)^{p+1+\tau} \frac{d^{(q-1+\tau)_+}(\zeta, F)}{d^{\min(p,r)}(\zeta, E)} \leq C_2 \cdot K.$$

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For this F, let $\beta(F)$ be its Minkowski type, *i.e.,*

$$\beta(F) = \sup\{\beta : m(F_s) = O(s^\beta), \ \ s \to 0+\},$$

and $F_s = \{t \in \mathbb{T} : d(t, F) < s\}$.

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Let $E, F \subset \mathbb{T}, \#E < \infty$, F be of Minkowski dimension $\beta(F)$. Let $\bar{F} \cap E = \emptyset$.

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$$\sum_{\zeta \in Z(f)} (1-|\zeta|)^{p+1+\tau} \frac{d^{(q-\beta(F)+\tau)_+}(\zeta,F)}{d^{\min(p,r)}(\zeta,E)} \le C_3 \cdot K.$$

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Problem : what happens when $\bar{F} \cap E \neq \emptyset$?

One will need to apply above theorems for special domains ($\mathbb{C}_+, \mathbb{C} \backslash \mathbb{R}_+$, etc.).

Zeros of holomorphic functions from different classes

Let $X' = \{x'_k\}_{k=1,\dots,\infty} \subset \mathbb{R}$ and $\beta(X')$ is the Minkowski dimension of the pre-image of X' on $\mathbb{T}$.

Theorem (BBGK)

Let $g \in Hol(\mathbb{C}_+)$, $|g(i)| = 1$ such that

$$\log|g(w)| \le \frac{K|w|^c(1+|w|)^b}{(\mathrm{Im}\ w)^a\, d(w,X')^d}, \quad a,b,d \ge 0, c \in \mathbb{R}.$$

Then for every $\tau > 0$ there exists a constant C_4 such that

$$\sum_{w\in Z(g)} \frac{(\mathrm{Im}\ w)^{a+1+\tau} d^{(d-\beta(X')+\tau)_+}(w,X')|w|^{(c)_{a,\tau}}}{(1+|w|)^{L_1}} \le C_4 K.$$

Zeros of holomorphic functions from different classes

Above,

$$(c)_{a,\tau} = (c_- - 1 + \tau)_+ - \min(c_+, a),$$

$$l = 2a + 2d - b - c = l_+ - l_-,$$

and

$$L_1 = 2(a + 1 + \tau) + 2(d - \beta(X') + \tau)_+ + (l_- - 1 + \tau)_+ + (c)_{a,\tau}.$$

Zeros of holomorphic functions from different classes

A similar result holds for $\Theta = \mathbb{C} \backslash \mathbb{R}_+$. Indeed, let $Y' = \{y'_k\}_{k=1,\dots,\infty} \subset \mathbb{R}_+$ and $\beta(Y')$ is the Minkowski dimension of the pre-image of Y' on $\mathbb{T}$.

Theorem (BBGK)

Let $h \in Hol(\Theta)$, $|h(-1)| = 1$ such that

$$\log |h(v)| \le \frac{K|v|^r (1+|v|)^{b/2}}{d^a(v, \mathbb{R}_+)\, d(v, Y')^d}, \quad a, b, d \ge 0, r \in \mathbb{R}.$$

Then for every $\tau > 0$ there exists a constant C_5 such that

$$\sum_{w \in Z(h)} \frac{d^{a+1+\tau}(v, \mathbb{R}_+)\, d^{(d-\beta(Y')+\tau)_+}(v, Y')}{|v|^{L_2/2}(1+|w|)^{L_3/2}} \le C_5 K.$$

Zeros of holomorphic functions from different classes

Above,

$$(c)_{a,\tau} = (c_- - 1 + \tau)_+ - \min(c_+, a), \quad c = 2r - a - d,$$

$$l = 3a + 3d - b - 2r = l_+ - l_-,$$

$$L_2 = (a + 1 + \tau) + (d - \beta(Y') + \tau)_+ - (c)_{a,\tau},$$

and

$$L_3 = 2(a + 1 + \tau) + 2(d - \beta(Y') + \tau)_+ + (c)_{a,\tau} - (l_- - 1 + \tau)_+.$$

Applications : Schrödinger operators on waveguides

- Briet-Kovarik-Raikov [2008], Briet-Kovarik-Raikov-Soccorsi [2013].

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Let $x = (x_\perp, x') \in \mathbb{R}^{d+1}$, and $x_\perp \in \Omega_0$, a bounded region in $\mathbb{R}^d$. For the moment, $\Omega = \Omega_0 \times \mathbb{R}$, a cylindrical domain (*i.e.,* a simplest model for a waveguide).

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Consider

$$H_{0\perp} f = (-\Delta_\perp) f, \quad \Gamma_\perp f\big|_{\partial\Omega_0} = 0, \qquad f \in L^2(\Omega_0).$$

The latter relation is a boundary condition (Dirichlet, von Neumann) for the operator.

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We suppose that $\lambda_j \asymp j^\alpha, \ \alpha > 0$, and $m_j \lesssim j^\beta, \ \beta \geq 0$.

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Furthemore, extend the boundary condition Γ from $\partial\Omega_0$ to $\partial\Omega, \Omega = \Omega_0 \times \mathbb{R}$ in x'-invariant way ; this boundary condition is also denoted Γ. Consider

$$H_0 f(x_\perp, x') = (-\Delta) f(x_\perp, x') = (-\Delta_\perp) \otimes \left(-\frac{d^2}{dx'^2} \right) f(x_\perp, x'),$$

and $\Gamma f\big|_{\partial\Omega} = 0$, where $f \in L^2(\Omega)$.

Applications : Schrödinger operators on waveguides

Of course, one has

$$(-\Delta) = \sum_j \lambda_j P_{\perp j} \otimes \left(-\frac{d^2}{dx'^2}\right),$$

or

$$(H_0 - \lambda)^{-1} = \sum_j P_{\perp j} \otimes \left(-\frac{d^2}{dx'^2} + \lambda_j - \lambda\right)^{-1}, \quad \lambda \notin \sigma(H_0),$$

where $\{P_{\perp j}\}$ are spectral projectors of $(-\Delta_\perp)$.

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By the way, $\sigma(H_0) = \mathbb{R}_{\lambda_0,+}$, where $\mathbb{R}_{\lambda_0,+} = [\lambda_0, +\infty)$.

Applications : Schrödinger operators on waveguides

Then, have a look at

$$Hf = (H_0 + V)f, \qquad f \in L^2(\Omega),$$

and $|V(x_\perp, x')| \le R(x_\perp)S(x')$ with $R \in L^\infty(\Omega_0)$ and $S \in L^p(\mathbb{R})$, $R, S \ge 0$.

Applications : Schrödinger operators on waveguides

The point is to apply the above theorems on zero distribution of holomorphic functions to

$$G(\lambda) = \det{}_p(I + V(H_0 - \lambda)^{-1}), \quad \lambda \in \Theta = \mathbb{C}\backslash\mathbb{R}_{\lambda_0,+}.$$

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Proposition

Under above hypotheses, let $p > \frac{\beta+1}{\alpha} + \frac{1}{2}$. Then

$$||V(H_0 - \lambda)^{-1}||^p_{\mathcal{S}_p} \leq C_6 ||R||^p_\infty ||S||^p_p \left(\sum_j \frac{m_j}{|\lambda - \lambda_j|^{1/2} d^{p-1}(\lambda, \mathbb{R}_{\lambda_j,+})} \right).$$

Applications : Schrödinger operators on waveguides

Proposition

For $\alpha > 0$, *one has*

$$||V(H_0 - \lambda)^{-1}||_{\mathcal{S}_p}^p \leq C_8 \frac{(1+|\lambda|)^{(\beta+1)/\alpha}}{d(\lambda,\Lambda)^{1/2} d^{p-1}(\lambda, \mathbb{R}_{\lambda_0,+})}, \quad \lambda \in \Theta.$$

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Theorem (BGGK)

Under above hypotheses and $p > \max(\frac{\beta+1}{\alpha} + \frac{1}{2}, 2)$ *, one has for every* $\tau > 0$

$$\sum_{v \in \sigma_d(H_0)} \frac{d^{p+\tau}(v, \mathbb{R}_{\lambda_0,+})\, d^{\frac{1}{2}((2-\alpha)/(2+\alpha)+2\tau)_+}(v, \Lambda)}{|v|^{L_2/2}(1+|w|)^{L_3/2}} \le C_9\, K.$$

Above, $L_2 = \frac{3}{2} + \frac{1}{2}(\frac{2-\alpha}{2+\alpha} + \tau)_+$, *and*

$$L_3 = 3\left(p - \frac{1}{2}\right) + \left(\frac{2-\alpha}{2+\alpha} + 2\tau\right)_+ + \tau.$$

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Thank you !