On Lieb-Thirring inequalities, zeros of holomorphic functions, and waveguides

S. Kupin (jointly with Ph. Briet, V. Bruneau, L. Golinskii)

IMB, Université de Bordeaux

Conference on Waveguides
May, 18, 2016
Plan of the talk

1. A general scheme.
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2. Zeros of holomorphic functions from different classes.
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3. Applications to Schrödinger operators on waveguides.
4. Some open problems.
Few references

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- **2013** – C. Dubuisson, D. Sambou, ...

- **2014** – R. Frank, J. Sabin, B. Simon, J.C. Cuenin, ...
A general scheme

Let $A_0 : H \rightarrow H$ be an operator on a Hilbert space, and $K : H \rightarrow H$ be an operator lying in $S_p$, $1 \leq p < \infty$. Recall that

$$S_p = \{ K \in S_\infty : ||K||_p^p := ||K||_{S_p}^p = \sum_k s_k(A)^p < \infty \},$$

where $s_k(K) = \lambda_k (K^*K)^{1/2}$.

Consider $A = A_0 + K$. One has $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(A_0)$.
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Problem: What can one say about $\sigma_{d} (A)$ and its distributional characteristics?
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A general scheme

One needs to look at the so-called regularised determinant. For $K \in S_p$ and $p \in \mathbb{N}^*$, define

$$\det_p(I + K) = \prod_k (1 + \lambda_k) \exp \left( \sum_{j=1}^{p-1} \frac{(-1)^j}{j} \chi^j_k \right),$$

where $\lambda_k = \lambda_k(K)$. 

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$$

where $\lambda_k = \lambda_k(K)$.

Furthermore, consider the regularised perturbation determinant, i.e.,

$$
G(\lambda) = \det_p(A - \lambda I)(A_0 - \lambda I)^{-1} = \det_p(I + K(A_0 - \lambda I)^{-1}).
$$
A general scheme

The properties of $G(.)$:

- $G \in Hol(\wp(A_0))$, where $\wp(A_0) = \bar{\mathbb{C}} \setminus \sigma(A_0)$,
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The properties of $G(.)$:

- $G \in Hol(\varrho(A_0))$, where $\varrho(A_0) = \bar{\mathbb{C}} \setminus \sigma(A_0)$,
- $Z(G)$, the zero set of $G$, coincides with $\sigma_d(A)$ up to multiplicities,
- there is a special bound of $G$ on $\varrho(A_0)$, i.e.,

$$\log |G(\lambda)| \leq \Gamma_p \|K(A_0 - \lambda)^{-1}\|_{S_p}^p \leq \Gamma_p \|K\|_{S_p}^p \|(A_0 - \lambda)^{-1}\|_{S_p}^p \leq \Gamma_p \frac{\|K\|_{S_p}^p}{d(\lambda, \sigma(A_0))^p}$$

with $\lambda \in \varrho(A_0)$. 

A general scheme
A general scheme

Now, let \( \varphi : \mathbb{D} = \{ z : |z| < 1 \} \to \varrho(A_0) \) and \( \psi : \varrho(A_0) \to \mathbb{D} \) be the conformal maps of the corresponding domains, \( \psi = \varphi^{-1} \). Make a “change of variables” \( \lambda = \varphi(z), \ z \in \mathbb{D} \).
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Now, let $\varphi : \mathbb{D} = \{ z : |z| < 1 \} \rightarrow \varrho(A_0)$ and $\psi : \varrho(A_0) \rightarrow \mathbb{D}$ be the conformal maps of the corresponding domains, $\psi = \varphi^{-1}$. Make a “change of variables” $\lambda = \varphi(z)$, $z \in \mathbb{D}$.

One comes to $g(z) = G(\varphi(z)) \in \text{Hol}(\mathbb{D})$ such that

$$\log |g(z)| \leq \frac{K}{d^p(z, \mathbb{T})} \frac{d^r(z, E)}{d^q(z, F)}, \quad z \in \mathbb{D}, \ p, q, r \geq 0,$$

and $E, F \subset \mathbb{T}$, $\mathbb{T} = \{ z : |z| = 1 \}$, $\#E, \#F < \infty$ and $E \cap F = \emptyset$. Of course,

$$d(z, E) = \inf_{t \in E} |z - t|,$$

so, for instance, $d(z, \mathbb{T}) = (1 - |z|)$, $z \in \mathbb{D}$. 
Zeros of holomorphic functions from different classes

- “Classical” Blaschke conditions.
Zeros of holomorphic functions from different classes

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- Let $F \subset \mathbb{T}$, $\# F < \infty$.

**Theorem (Borichev-Golinskii-K’ 2009)**

Let $f \in Hol(\mathbb{D})$, $|f(0)| = 1$, satisfy the growth condition

$$
\log |f(z)| \leq \frac{K}{(1 - |z|)^p \, d^q(z, F)}
$$

for $z \in \mathbb{D}$ and $p, q \geq 0$. Then for each $\tau > 0$ there is a positive constant $C_1 = C_1(p, q, F; \tau)$ such that

$$
\sum_{\zeta \in Z(f)} (1 - |\zeta|)^{p+1+\tau} \, d^{q-1+\tau}(\zeta, F) \leq C_1 \cdot K.
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where $z \in \mathbb{D}$ and $p, q, r \geq 0$. Then for each $\tau > 0$, there is a positive constant $C_2 = C_2(p, q, r, E, F; \tau)$ such that

$$ \sum_{\zeta \in Z(f)} (1 - |\zeta|)^{p+1+\tau} \frac{d^{q-1+\tau}(\zeta, F)}{d^\min(p,r)(\zeta, E)} \leq C_2 \cdot K. $$
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Zeros of holomorphic functions from different classes

What happens when for any (measurable) $F \subset \mathbb{T}$?

For this $F$, let $\beta(F)$ be its Minkowski type, i.e.,

$$\beta(F) = \sup\{\beta : m(F_s) = O(s^\beta), \ s \to 0^+\},$$

and $F_s = \{t \in \mathbb{T} : d(t, F) < s\}$. 

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Zeros of holomorphic functions from different classes

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$$\sum_{\zeta \in Z(f)} (1 - |\zeta|)^{p+1+\tau} \frac{d^{q-\beta(F)+\tau+}(\zeta, F)}{d^{\min(p,r)}(\zeta, E)} \leq C_3 \cdot K.$$
Zeros of holomorphic functions from different classes

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**Problem**: what happens when $\bar{F} \cap E \neq \emptyset$?

One will need to apply above theorems for special domains ($\mathbb{C}_+, \mathbb{C}\setminus\mathbb{R}_+, \text{etc.}$).
Zeros of holomorphic functions from different classes

Let $X' = \{x'_k\}_{k=1,...,\infty} \subset \mathbb{R}$ and $\beta(X')$ is the Minkowski dimension of the pre-image of $X'$ on $\mathbb{T}$.

Theorem (BBGK)

Let $g \in Hol(\mathbb{C}_+)$, $|g(i)| = 1$ such that

$$
\log |g(w)| \leq \frac{K|w|^c(1 + |w|)^b}{(\text{Im } w)^a d(w, X')^d}, \quad a, b, d \geq 0, c \in \mathbb{R}.
$$

Then for every $\tau > 0$ there exists a constant $C_4$ such that

$$
\sum_{w \in \mathbb{Z}(g)} \frac{(\text{Im } w)^{a+1+\tau} d^{(d-\beta(X')-\tau)+} (w, X') |w|^{(c)_{a,\tau}}}{(1 + |w|)^{L_1}} \leq C_4 K.
$$
Zeros of holomorphic functions from different classes

Above,

\[(c)_{a,\tau} = (c_- - 1 + \tau)_+ - \min(c_+, a),\]

\[l = 2a + 2d - b - c = l_+ - l_-,\]

and

\[L_1 = 2(a + 1 + \tau) + 2(d - \beta(X') + \tau)_+ + (l_- - 1 + \tau)_+ + (c)_{a,\tau}.\]
Zeros of holomorphic functions from different classes

A similar result holds for $\Theta = \mathbb{C}\setminus\mathbb{R}_+$. Indeed, let $Y' = \{y'_k\}_{k=1,\ldots,\infty} \subset \mathbb{R}_+$ and $\beta(Y')$ is the Minkowski dimension of the pre-image of $Y'$ on $\mathbb{T}$.

Theorem (BBGK)

Let $h \in \text{Hol}(\Theta)$, $|h(-1)| = 1$ such that

$$\log |h(v)| \leq \frac{K|v|^r(1 + |v|)^{b/2}}{d^a(v, \mathbb{R}_+) d(v, Y')^d}, \quad a, b, d \geq 0, r \in \mathbb{R}.$$ 

Then for every $\tau > 0$ there exists a constant $C_5$ such that

$$\sum_{w \in Z(h)} \frac{d^{a+1+\tau}(v, \mathbb{R}_+) d^{(d-\beta(Y')-\tau)+}(v, Y')}{|v|^{L_2/2}(1 + |w|)^{L_3/2}} \leq C_5 K.$$
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Above,

\[ (c)_{a,\tau} = (c_- - 1 + \tau)_+ - \min(c_+, a), \quad c = 2r - a - d, \]

\[ l = 3a + 3d - b - 2r = l_+ - l_- , \]

\[ L_2 = (a + 1 + \tau) + (d - \beta(Y') + \tau)_+ - (c)_{a,\tau}, \]

and

\[ L_3 = 2(a + 1 + \tau) + 2(d - \beta(Y') + \tau)_+ + (c)_{a,\tau} - (l_- - 1 + \tau)_+. \]
Applications: Schrödinger operators on waveguides

- Briet-Kovarik-Raikov [2008], Briet-Kovarik-Raikov-Soccorsi [2013].

\[
\begin{align*}
x &= (x_\perp, x') \\
\Omega_0 &= \Omega \times \mathbb{R}, \\
H_{\perp} f &= (-\Delta_\perp) f, \\
\Gamma_\perp f |_{\partial \Omega_0} &= 0, \\
\lambda_j &\rightarrow +\infty,
\end{align*}
\]

It is well known that

\[
\Lambda := \sigma(H_{\perp}) = \{\lambda_j \}_{j=0}^{\infty},
\]

\[
\lambda_j \sim j^{2/d},
\]

\[
m_j \sim j^{d-1}.
\]
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Let $x = (x_\perp, x') \in \mathbb{R}^{d+1}$, and $x_\perp \in \Omega_0$, a bounded region in $\mathbb{R}^d$. For the moment, $\Omega = \Omega_0 \times \mathbb{R}$, a cylindrical domain (i.e., a simplest model for a waveguide).
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Consider

\[
H_0 \perp f = (-\Delta_\perp) f, \quad \Gamma_\perp f|_{\partial \Omega_0} = 0, \quad f \in L^2(\Omega_0).
\]

The latter relation is a boundary condition (Dirichlet, von Neumann) for the operator.
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- \( j^{2/d} \lesssim \lambda_j \lesssim j^2 \), \quad 1 \leq m_j \lesssim j^{d-1}. \)
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We suppose that $\lambda_j \asymp j^\alpha$, $\alpha > 0$, and $m_j \lesssim j^\beta$, $\beta \geq 0$. 

Guess: similar bounds are true when $\lambda_j \simeq j^\alpha$.

Problem: give examples of regions $\Omega_0$ providing prescribed behavior for $\{\lambda_j\}, \{m_j\}$.

Furthermore, extend the boundary condition $\Gamma$ from $\partial \Omega_0$ to $\partial \Omega$, $\Omega = \Omega_0 \times \mathbb{R}$ in $x'$-invariant way; this boundary condition is also denoted $\Gamma$. Consider $H_0 f(x_\perp, x') = (-\Delta) f(x_\perp, x') = (-\Delta_\perp) \otimes (-d^2/dx'^2) f(x_\perp, x')$, and $\Gamma f \mid_{\partial \Omega} = 0$, where $f \in L^2(\Omega)$. 
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and \( \Gamma f|_{\partial \Omega} = 0 \), where \( f \in L^2(\Omega) \).
Applications : Schrödinger operators on waveguides

Of course, one has

\[(\Delta) = \sum_{j} \lambda_{j} P_{\perp j} \otimes \left(-\frac{d^{2}}{dx'^{2}}\right)\, ,\]

or

\[(H_{0} - \lambda)^{-1} = \sum_{j} P_{\perp j} \otimes \left(-\frac{d^{2}}{dx'^{2}} + \lambda_{j} - \lambda\right)^{-1}\, , \quad \lambda \notin \sigma(H_{0}),\]

where \(\{P_{\perp j}\}\) are spectral projectors of \((-\Delta_{\perp})\).
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where \( \{P_{\perp j}\} \) are spectral projectors of \( (-\Delta_{\perp}) \).

By the way, \( \sigma(H_0) = \mathbb{R}_{\lambda_0,+} \), where \( \mathbb{R}_{\lambda_0,+} = [\lambda_0, +\infty) \).
Then, have a look at

\[ Hf = (H_0 + V)f, \quad f \in L^2(\Omega), \]

and \(|V(x_\perp, x')| \leq R(x_\perp)S(x')\) with \(R \in L^\infty(\Omega_0)\) and \(S \in L^p(\mathbb{R})\), \(R, S \geq 0\).
The point is to apply the above theorems on zero distribution of holomorphic functions to

\[ G(\lambda) = \det_{\rho}(I + V(H_0 - \lambda)^{-1}), \quad \lambda \in \Theta = \mathbb{C} \setminus \mathbb{R}_{\lambda_0,+}. \]
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The point is to apply the above theorems on zero distribution of holomorphic functions to

\[ G(\lambda) = \det_p(I + V(H_0 - \lambda)^{-1}), \quad \lambda \in \Theta = \mathbb{C} \setminus \mathbb{R}_{\lambda_0,+}. \]

**Proposition**

*Under above hypotheses, let* \( p > \frac{\beta + 1}{\alpha} + \frac{1}{2} \). *Then*

\[
\|V(H_0 - \lambda)^{-1}\|_{S_p}^p \leq C_6 \|R\|_{\infty}^p \|S\|_p^p \left( \sum_j \frac{m_j}{|\lambda - \lambda_j|^{1/2}} d^{p-1}(\lambda, \mathbb{R}_{\lambda_j,+}) \right).
\]
Applications: Schrödinger operators on waveguides

**Proposition**

For $\alpha > 0$, one has

$\| V(H_0 - \lambda)^{-1} \|_{S_p}^p \leq C_8 \frac{(1 + |\lambda|)^{(\beta + 1)/\alpha}}{d(\lambda, \Lambda)^{1/2} d^{p-1}(\lambda, \mathbb{R}_{\lambda_0,+})}$, \( \lambda \in \Theta \).
For $\Lambda = \{\lambda_j\}_{j=1,\ldots,\infty}$, $j \sim j^\alpha$, one has $\beta(\Lambda) = \alpha / (\alpha + 2)$.
Applications: Schrödinger operators on waveguides

For $\Lambda = \{\lambda_j\}_{j=1,\ldots,\infty}, j \sim j^\alpha$, one has $\beta(\Lambda) = \alpha / (\alpha + 2)$.

**Theorem (BGGK)**

*Under above hypotheses and $p > \max\left(\frac{\beta + 1}{\alpha}, \frac{1}{2}, 2\right)$, one has for every $\tau > 0$*

$$
\sum_{v \in \sigma_d(H_0)} d^{p+\tau}(v, \mathbb{R}_{\lambda_0, +}) \frac{d^{1/2}((2-\alpha)/(2+\alpha)+2\tau)+(v, \Lambda)}{|v| L_2/2 (1 + |w|) L_3/2} \leq C_9 K.
$$

*Above, $L_2 = \frac{3}{2} + \frac{1}{2} \left(\frac{2-\alpha}{2+\alpha} + \tau\right)_+$, and*

$$
L_3 = 3 \left(p - \frac{1}{2}\right) + \left(\frac{2-\alpha}{2+\alpha} + 2\tau\right)_+ + \tau.
$$
Some open problems

- The case of Robin boundary conditions on $\partial \Omega_0$ (or on $\partial \Omega$),

- The case of a twisted waveguide; even the model of a constant twist is interesting,

- More general kinds of waveguides (bent waveguides, bent and twisted waveguides, etc.).
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Thank you!