Absence of trapped modes for a Y-shaped junction of open waveguides

Christophe Hazard

with Anne-Sophie Bonnet-Ben Dhia and Sonia Fliss

POEMS (Propagation d'Ondes: Etude Mathématique et Simulation)
CNRS/ENSTA-ParisTech/INRIA, Palaiseau, France

Porquerolles – May 2016

The story began long time ago...

when Yves explained to Anne-Sophie the proof of Ricardo...*

Since then, we explore the idea...

*: Weder, *Absence of eigenvalues of the acoustic propagator in deformed wave guides.* Rocky Mountain J. Math. 18 (1988)

Outline

1. Trapped modes in waveguides
2. A Rellich type theorem in a homogeneous medium
3. Y-shaped junction of waveguides

Outline

1. Trapped modes in waveguides
2. A Rellich type theorem in a homogeneous medium
3. Y-shaped junction of waveguides

What are “trapped modes” ?

Time-harmonic self-existing oscillations of a propagative medium which are localized in space (L^2).

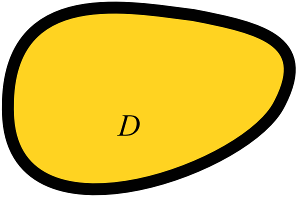
Our model: acoustic media described by the Helmholtz equation

$$\begin{cases} \Delta u + \omega^2 n^2 \, u = 0 \quad \text{in } D \subset \mathbb{R}^d \\ + \text{ non-dissipative b.c. if } \partial D \neq \emptyset, \end{cases}$$

with $\omega \in \mathbb{R}$ and $n = n(x) \in \mathbb{R}$.

QUESTION : For given D and n, can one find $(\omega, u) \in \mathbb{R} \times L^2(D) \setminus \{0\}$ solution to the above problem?

The case of a bounded cavity D

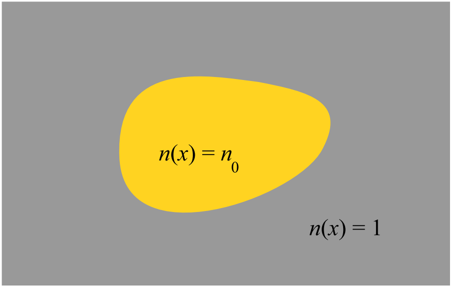


$\Longrightarrow$ infinite sequence of trapped modes

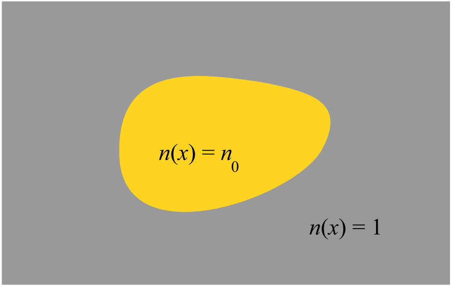
$\Longrightarrow$ eigenfrequencies $\omega_n \to +\infty$ (= discrete spectrum of $n^{-2}\Delta$)

What about unbounded domains D?

Opening the cavity: immersion in a homogeneous medium



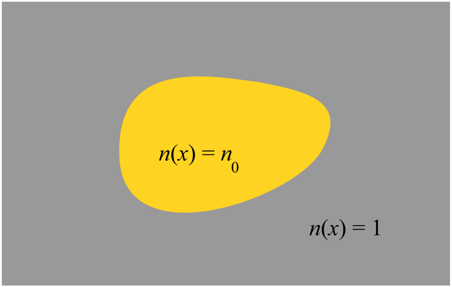
Opening the cavity: immersion in a homogeneous medium



Rellich (1943) uniqueness theorem : for $R > 0$,

$$\text{if } u \in L^2(|x| > R) \text{ satisfies } \Delta u + \omega^2 u = 0 \text{ in } |x| > R, \text{ then } u \equiv 0.$$

Opening the cavity: immersion in a homogeneous medium

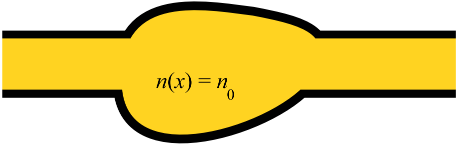


Rellich (1943) uniqueness theorem + unique continuation

$\Longrightarrow$ no trapped modes

$\Longrightarrow$ no eigenvalue embedded in the continuous spectrum $\mathbb{R}^+$ of $n^{-2}\Delta$.

Opening the cavity: closed waveguide

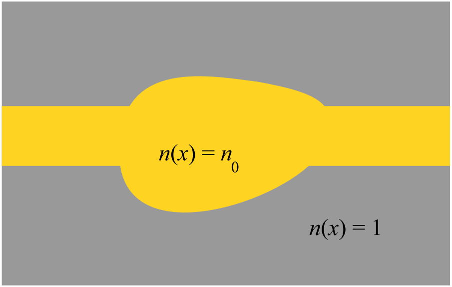


$\Longrightarrow$ trapped modes may occur

$\Longrightarrow$ at most a discrete set of eigenfrequencies, embedded or not in the continuous spectrum of $-n^{-2}\Delta$

Evans, Levitin and Vassiliev, Witsch, Linton and McIver, Nazarov...

Opening the cavity: open waveguide



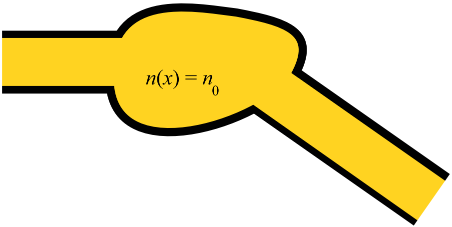
$\Longrightarrow$ no more trapped modes

$\Longrightarrow$ no eigenvalue embedded in the continuous spectrum $\mathbb{R}^+$ of $-n^{-2}\Delta$

Weder (1991), DeBièvre and Pravica (1992)

Bonnet-Ben Dhia *et al.* (2009), H. (2014)

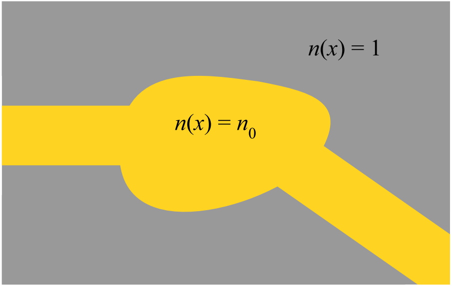
What about a bended waveguide ?



$\Longrightarrow$ trapped modes generally occur

Duclos and Exner (1995), Krejcirik, Freitas, Dauge and Raymond, ...

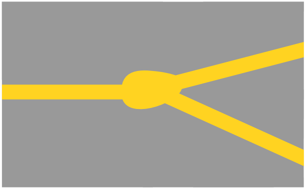
Open bended waveguide



$\Longrightarrow$ no trapped modes

Bonnet-Ben Dhia, Fliss, H., Tonnoir (2016) ▷ part 2 of the talk

Multiple junctions of waveguides



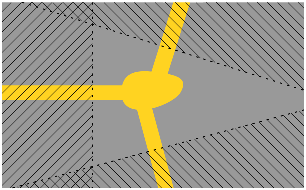
No trapped modes

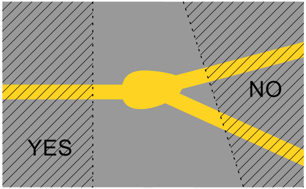
▷ part 3 of the talk

Nobody knows, but...

The reason for that is that

we use Fourier representations in half-planes:





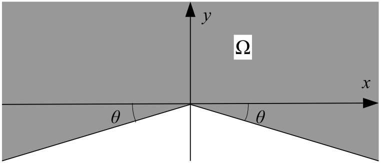
$\Longrightarrow$ OK if all angles between branches are greater than $\pi/2$

Outline

1. Trapped modes in waveguides
2. A Rellich type theorem in a homogeneous medium
3. Y-shaped junction of waveguides

A simple statement with a simple proof

Let $\theta \in (0, \pi/2)$ and $\Omega := \left\{(x,y) \in \mathbb{R}^2 \text{ such that } y > -|x| \tan\theta\right\}$:



Theorem (Bonnet-Ben Dhia, Fliss, H., Tonnoir (2016))

If $u \in L^2(\Omega)$ satisfies $\Delta u + \omega^2 u = 0$ in Ω, then $u \equiv 0$.

- No boundary condition: Rellich type theorem
- Optimal (false if $\theta = 0$)
- Simple proof (elementary tools)

Consequences for trapped modes

This theorem, combined with a unique continuation principle, proves that there are no trapped modes (= no embedded eigenvalues) in the following configurations:

General ideas of the proof

Two main ingredients (Weder (1988)):

1. First (easy) step: introduce $\widehat{u}$ the partial Fourier transform of u in x

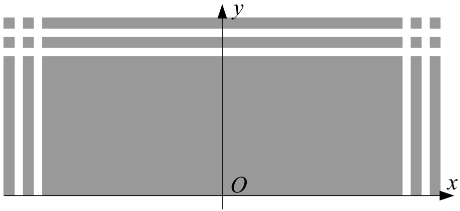
$$\widehat{u}(\xi, y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x, y)\, \mathrm{e}^{-\mathrm{i}x\xi}\, \mathrm{d}x$$

   and use an energy argument to prove that $\widehat{u}(\xi, 0) = 0$ for $|\xi| < \omega$.

2. Second step: prove that $\xi \longmapsto \widehat{u}(\xi, 0)$ is analytic in a vicinity of the real axis, and conclude that $\widehat{u}(\xi, 0) \equiv 0$ for all $\xi \in \mathbb{R}$, which implies that $u \equiv 0$ in Ω.

1st (easy) step : Fourier representation in a half-plane

Consider the upper half-plane $y > 0$:



- As $u \in L^2(\Omega)$ satisfies $\Delta u + \omega^2 u = 0$ in Ω, for a.e. $\xi \in \mathbb{R}$, function $\widehat{u}(\xi, \cdot)$ belongs to $L^2(y > 0)$ and satisfies

$$\frac{\mathrm{d}^2 \widehat{u}}{\mathrm{d}y^2} + (\omega^2 - \xi^2)\widehat{u} = 0 \quad \text{for } y > 0.$$

- Since $\widehat{u}(\xi, \cdot) \in L^2(y > 0)$:

$$|\xi| > \omega \implies \widehat{u}(\xi, y) = \widehat{u}(\xi, 0) e^{-\sqrt{\xi^2 - \omega^2}\, y}$$

$$|\xi| < \omega \implies \widehat{u}(\xi, y) = 0$$

1st (easy) step : Fourier representation in a half-plane

Using the inverse Fourier transform, we obtain

Fourier representation of u :

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{|\xi|>\omega} \widehat{u}(\xi,0)\, \boxed{\mathrm{e}^{-\sqrt{\xi^2-\omega^2}\, y}\, \mathrm{e}^{\mathrm{i}\xi x}}\, \mathrm{d}\xi \quad \text{for } x \in \mathbb{R} \text{ and } y > 0.$$

This is a modal representation of u (superposition of y-evanescent modes)

Consequence of the finite energy (L^2) assumption

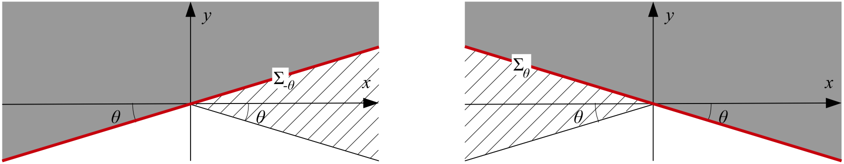
$\widehat{u}(\xi,0) = 0$ for $|\xi| < \omega \quad \Longleftrightarrow \quad$ no propagative components

2nd step : analyticity

The idea is to write

$$\widehat{u}(\xi, 0) = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} \boxed{u(x,0)} \, \mathrm{e}^{-\mathrm{i}x\xi} \, \mathrm{d}x + \int_{0}^{+\infty} \boxed{u(x,0)} \, \mathrm{e}^{-\mathrm{i}x\xi} \, \mathrm{d}x \right)$$

and to express $\boxed{u(x,0)}$ using the previous Fourier representations in both following half-planes:



For instance, for $x > 0$

$$u(x,0) = \frac{1}{\sqrt{2\pi}} \int_{|\eta|>\omega} \widehat{\varphi}^{+}(\eta) \, \mathrm{e}^{-\sqrt{\eta^2-\omega^2}\,\sin\theta\, x} \, \mathrm{e}^{\mathrm{i}\eta\cos\theta\, x} \, \mathrm{d}\eta$$

where $\widehat{\varphi}^{+}$ denotes the Fourier transform of $u_{|\Sigma_\theta}$.

We get the following expression:

$$\widehat{u}(\xi,0)=\frac{1}{2\pi}\sum_{\pm}\int_{\mathbb{R}^\pm}\int_{|\eta|>\omega}\widehat{\varphi}^{\pm}(\eta)\,\mathrm{e}^{\boxed{x}\,(\mathrm{i}\eta\cos\theta\mp\sqrt{\eta^2-\omega^2}\sin\theta)}\,\mathrm{d}\eta\,\mathrm{e}^{-\mathrm{i}\,\boxed{x}\,\xi}\,\mathrm{d}\,\boxed{x}$$

where $\widehat{\varphi}^{\pm}$ denotes the Fourier transform of $u_{|\Sigma_{\pm\theta}}$.

By Fubini's theorem and explicit integration in $\boxed{x}$, we get finally:

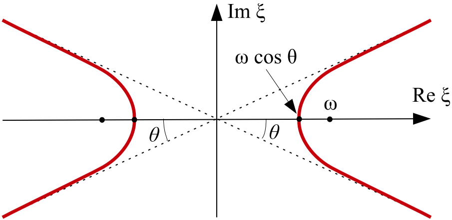
$$\widehat{u}(\xi,0)=\frac{1}{2\pi}\sum_{\pm}\int_{|\eta|>\omega}\frac{\widehat{\varphi}^{\pm}(\eta)}{\mp\mathrm{i}(\eta\cos\theta-\xi)+\sqrt{\eta^2-\omega^2}\sin\theta}\,\mathrm{d}\eta$$

Is it an analytic function of ξ?

By Lebesgue's dominated convergence theorem, the function

$$\widehat{u}(\xi,0) = \frac{1}{2\pi}\sum_{\pm}\int_{|\eta|>\omega} \frac{\widehat{\varphi}^{\pm}(\eta)}{\mp \mathrm{i}(\eta\cos\theta - \xi) + \sqrt{\eta^2-\omega^2}\sin\theta}\,\mathrm{d}\eta$$

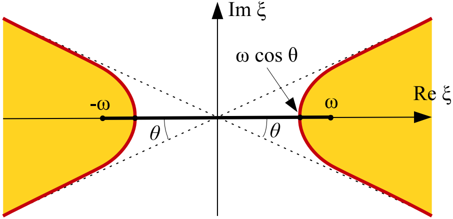
is analytic in ξ outside the hyperbola below.



Proof: just notice that the denominator vanishes for some η if and only if ξ belongs to the hyperbola.

End of the proof

- $\widehat{u}(\xi, 0)$ is analytic on the yellow domain
- it vanishes on the segment $(-\omega, +\omega)$ (1st step)



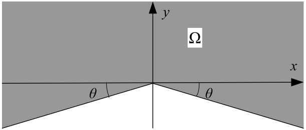
$\Longrightarrow$ $\widehat{u}(\xi, 0) = 0$ for all $\xi \in \mathbb{R}$ (isolated zeros of an analytic function)

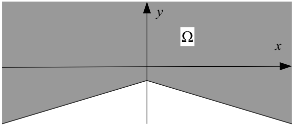
$\Longrightarrow$ $u(x, y) = 0$ in the half-plane $\mathbb{R} \times \mathbb{R}^+$ (Fourier representation)

$\Longrightarrow$ $\boxed{u \equiv 0 \text{ in } \Omega}$ (unique continuation)

A slight correction...

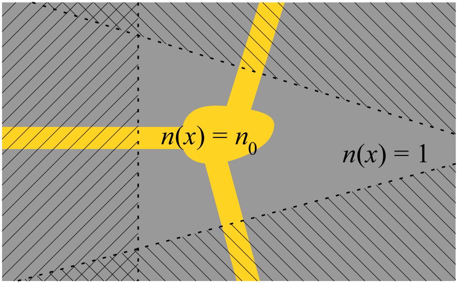
To make this proof correct, one just has to make at the beginning a translation of the x axis:





This provides the regularity of $u(\cdot, 0)$, and therefore the decay of $\widehat{u}(\cdot, 0)$, which are required to apply Fubini and Lebesgue theorems.

Absence of trapped modes

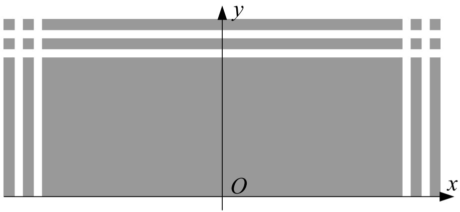


Theorem (Bonnet-Ben Dhia, Fliss, H. (some weeks ago))

If $u \in L^2(\mathbb{R}^2)$ satisfies $\Delta u + n^2\omega^2 u = 0$ in $\mathbb{R}^2$, then $u \equiv 0$.

$\Longrightarrow$ Basic tool : use *generalized* Fourier representations in the 3 hatched half-planes (see Julian Ott's talk, Thursday 9:00).

Back to $\mathcal{F}$ourier representation in a half-plane



If $u \in L^2$ satisfies $-\Delta u - \omega^2 u = 0$ in $\{y > 0\}$, then

$$u(x,y) = \int_{|\xi|>\omega} \mathcal{F}u(\xi,0)\, \boxed{e^{-\sqrt{\xi^2-\omega^2}\,y}\, \frac{e^{i\xi x}}{\sqrt{2\pi}}}\, d\xi \quad \text{and} \quad \mathcal{F}u(\xi,0) = 0 \text{ if } |\xi| < \omega.$$

To obtain this representation, write

$$-\Delta u - \omega^2 u = \underbrace{\left(-\frac{d^2}{dx^2} - \omega^2\right)}_{A} u - \frac{d^2 u}{dy^2} \quad \text{and} \quad \boxed{\text{diagonalize } A.}$$

Back to $\mathcal{F}$ourier representation in a half-plane

- $\mathcal{F}$ diagonalizes the operator $A = -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} - \omega^2$ in the sense that

$$\mathcal{F} A\, \varphi(\xi) = \lambda_\xi\ \mathcal{F}\, \varphi(\xi) \quad \text{where} \quad \lambda_\xi = \xi^2 - \omega^2.$$

- A is selfadjoint in $L^2(\mathbb{R})$ with purely continuous spectrum $[-\omega^2, +\infty[$.

- $\mathcal{F}$ appears as an operator of decomposition on a family of *generalized* eigenfunctions $\Phi_\xi \notin L^2(\mathbb{R})$ of A :

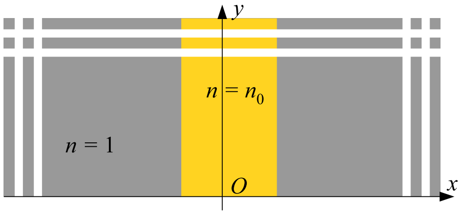
$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}} \varphi(x)\, \overline{\Phi_\xi(x)}\, \mathrm{d}x \quad \text{where} \quad \Phi_\xi(x) = \frac{\mathrm{e}^{\mathrm{i}x\xi}}{\sqrt{2\pi}} \text{ satisfies } A\, \Phi_\xi = \lambda_\xi\ \Phi_\xi.$$

- $\mathcal{F}^{-1}$ is the operator of re-composition on this family:

$$\mathcal{F}^{-1}\widehat{\varphi}(x) = \int_{\mathbb{R}} \widehat{\varphi}(\xi)\, \Phi_\xi(x)\, \mathrm{d}\xi.$$

Generalized $\widetilde{\mathcal{F}}$ourier representation in a half-guide

Suppose $u \in L^2$ satisfies $\Delta u + \boxed{n^2(x)\,\omega^2}\, u = 0$ in $\{y > 0\}$.



Write

$$-\Delta u - n^2\omega^2\, u = \underbrace{\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - n^2\omega^2\right)}_{\widetilde{A}} u - \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} \quad \text{and} \quad \boxed{\text{diagonalize } \widetilde{A}\ ?}$$

$\Longrightarrow$ The *generalized* Fourier transform $\widetilde{\mathcal{F}}$ diagonalizes the operator $\widetilde{A}$ in the sense that

$$\widetilde{\mathcal{F}}\widetilde{A}\,\varphi(\xi) = \lambda_\xi\ \widetilde{\mathcal{F}}\,\varphi(\xi) \quad \text{where} \quad \lambda_\xi = \xi^2 - \omega^2.$$

Generalized $\widetilde{\mathcal{F}}$ourier representation in a half-guide

$\widetilde{A} = -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} - n^2\omega^2$ is selfadjoint in $L^2(\mathbb{R})$. Its spectrum is composed of

- a continuous spectrum $\Lambda_{\mathrm{c}} = [-\omega^2, +\infty[$ (same as A),
- a finite point spectrum $\Lambda_{\mathrm{p}} \subset]-\infty, -\omega^2[$ (nonempty iff $\sup n(x) > 1$).

Construction of a complete spectral family $\left\{\widetilde{\Phi}_\xi;\ \xi \in \mathbb{R} \cup \mathbb{G}\right\}$:

- *Generalized* eigenfunctions for $\xi \in \mathbb{R}$ (i.e., $\lambda_\xi = \xi^2 - \omega^2 \in \Lambda_{\mathrm{c}}$) :

$$\underbrace{\widetilde{\Phi}_\xi = \Phi_\xi + \Phi_\xi^{\mathrm{scat}} \notin L^2(\mathbb{R})}_{\text{radiation modes}} \quad \text{where} \quad \begin{cases} \widetilde{A}\,\widetilde{\Phi}_\xi = \lambda_\xi\,\widetilde{\Phi}_\xi \\ \Phi_\xi^{\mathrm{scat}}(x) = \alpha_\xi^{\pm}\,\mathrm{e}^{\mathrm{i}|\xi||x|} \text{ if } x \to \pm\infty. \end{cases}$$

- Eigenfunctions for $\xi \in \mathbb{G}$ = finite subset of $\mathrm{i}\mathbb{R}^+$ (i.e., for each $\lambda_\xi \in \Lambda_{\mathrm{p}}$):

$$\underbrace{\widetilde{\Phi}_\xi \in L^2(\mathbb{R})}_{\text{guided modes}} \quad \text{where} \quad \begin{cases} \widetilde{A}\,\widetilde{\Phi}_\xi = \lambda_\xi\,\widetilde{\Phi}_\xi \\ \|\widetilde{\Phi}_\xi\|_{L^2(\mathbb{R})} = 1. \end{cases}$$

Generalized $\widetilde{\mathcal{F}}$ourier representation in a half-guide

Spectral theory yields

- The operator of decomposition on the family $\{\widetilde{\Phi}_\xi;\ \xi \in \mathbb{R} \cup \mathbb{G}\}$:

$$\widetilde{\mathcal{F}}\varphi(\xi) := \int_{\mathbb{R}} \varphi(x)\, \widetilde{\Phi}_\xi(x)\, \mathrm{d}x \quad \forall \xi \in \mathbb{R} \cup \mathbb{G},$$

  extends to a unitary transformation from $L^2(\mathbb{R}_x)$ to $L^2(\mathbb{R}_\xi) \oplus \ell^2(\mathbb{G})$.

- $\widetilde{\mathcal{F}}^{-1} = \mathcal{F}^*$ is the operator of re-composition on the family $\{\widetilde{\Phi}_\xi\}$:

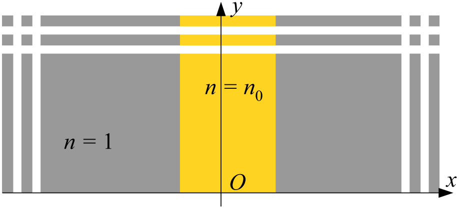
$$\widetilde{\mathcal{F}}^{-1}\widehat{\varphi} = \int_{\mathbb{R}} \widehat{\varphi}(\xi)\, \widetilde{\Phi}_\xi\, \mathrm{d}\xi + \sum_{\xi \in \mathbb{G}} \widehat{\varphi}(\xi)\, \widetilde{\Phi}_\xi.$$

- $\widetilde{\mathcal{F}}$ diagonalizes $\widetilde{A}$ in the sense that $\widetilde{A} = \widetilde{\mathcal{F}}^{-1} \lambda_\xi\, \widetilde{\mathcal{F}}$.

Moreover

For all $x \in \mathbb{R}$, function $\xi \mapsto \widetilde{\Phi}_\xi(x)$ extends to a meromorphic function of $\xi \in \mathbb{C}$.

Generalized $\widetilde{\mathcal{F}}$ourier representation in a half-guide

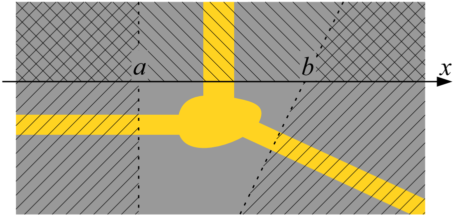


If $u \in L^2$ satisfies $\Delta u + n^2(x)\omega^2\, u = 0$ in $\{y > 0\}$, then

$$u(x,y) = \int_{|\xi|>\omega} \widetilde{\mathcal{F}}u(\xi,0)\ \boxed{\mathrm{e}^{-\sqrt{\xi^2-\omega^2}\,y}\ \widetilde{\Phi}_\xi}\ \mathrm{d}\xi$$

and $\widetilde{\mathcal{F}}u(\xi,0) = 0$ if $\xi \in\ ]-\omega,+\omega[\ \cup\ \mathbb{G}$.

Proof of the absence of trapped modes



The *generalized* $\widetilde{\mathcal{F}}$ourier representation tells us that

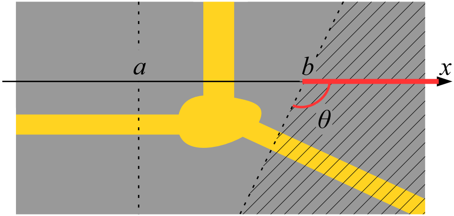
$$\widehat{u}(\xi) := \int_{\mathbb{R}} u(x,0)\, \widetilde{\Phi}_\xi(x)\, \mathrm{d}x \quad \text{vanishes if } \xi \in \,]-\omega, +\omega[$$

It remains to prove that $\xi \longmapsto \widehat{u}(\xi)$ is analytic in a vicinity of the real axis.
The idea:

$$\widehat{u}(\xi) = \int_{-\infty}^{a} \boxed{u(x,0)}\, \widetilde{\Phi}_\xi(x)\, \mathrm{d}x + \int_a^b u(x,0)\, \widetilde{\Phi}_\xi(x)\, \mathrm{d}x + \int_b^{+\infty} \boxed{u(x,0)}\, \widetilde{\Phi}_\xi(x)\, \mathrm{d}x$$

Proof of the absence of trapped modes

- $\displaystyle\int_a^b u(x,0)\,\widetilde{\Phi}_\xi(x)\,\mathrm{d}x$ is analytic near $\mathbb{R}$ since $\left\{\begin{array}{l}\xi \mapsto \widetilde{\Phi}_\xi(x) \text{ meromorphic,}\\ [a,b] \text{ bounded.}\end{array}\right.$



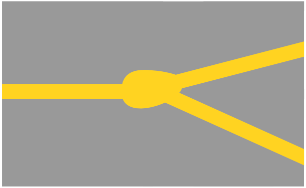
- $\displaystyle\int_b^{+\infty} \boxed{u(x,0)}\,\widetilde{\Phi}_\xi(x)\,\mathrm{d}x \implies$ use the GFR in the right (R) half-guide:

$$\boxed{u(x,0)} = \int_{|\eta|>\omega} \widehat{\varphi}^{(R)}(\eta)\,\mathrm{e}^{-\sqrt{\eta^2-\omega^2}\,x\,\sin\theta}\,\widetilde{\Phi}_\eta^{(R)}(x\,\cos\theta)\,\mathrm{d}\eta$$

and proceed as in §2. Recall that $\widetilde{\Phi}_\xi(x) = \dfrac{\mathrm{e}^{\mathrm{i}\xi x}}{\sqrt{2\pi}} + \alpha_\xi^\pm\,\mathrm{e}^{\mathrm{i}|\xi||x|}$ if $x \to \pm\infty$.

Conclusion: open questions

- 2D multiple junctions with angle $< \pi/2$:



- 3D multiple junctions
- scattering by junctions
- periodic waveguides

• • •

THANK YOU

for your attention