Spectral asymptotics for an elastic strip with an interior crack

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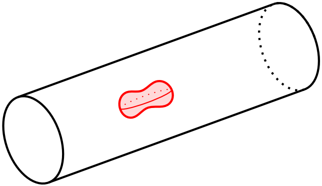
General Subject:

- Elastic media with a (small) perturbation.



Subject of the present talk:

- Existence of trapped modes

$\cong$ harmonic oscillation near the perturbation

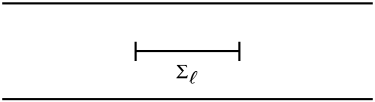
$\cong$ (embedded) eigenvalues of a suitable differential operator.

Applications:

- Non-destructive testing theory (wings of airplanes or sensitive structures).

General setting

Let $\Omega := \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and denote by $\Sigma_\ell = (-\ell, \ell)$ the crack. Let $\Omega_\ell := \Omega \setminus (\overline{\Sigma_\ell} \times \{0\})$. We consider the elasticity operator in $L_2(\Omega; \mathbb{C}^2)$ and traction free boundary conditions on $\partial\Omega_\ell$.



The operator acts as

$$A_{\Sigma_\ell} = -\mu\Delta - (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}$$

on functions $u \in H^1(\Omega_\ell; \mathbb{C}^2)$ such that

$$(\lambda\,\mathrm{div}\,u + 2\mu E(u)) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_\ell.$$

Here μ and λ are the Lamé constants and $E(u) = (\partial_i u_j + \partial_j u_i)_{i,j=1,2}$ is the strain of the elastic material.

- Existence of (embedded) eigenvalues?
- Asymptotic behaviour as $\ell \to 0$?

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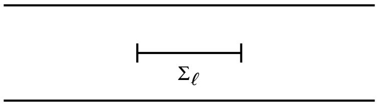

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- Existence of (embedded) eigenvalues? → [H., Schulz, Wirth].
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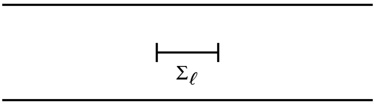

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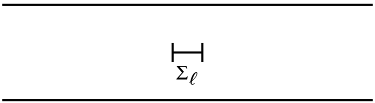

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Main result

Theorem

Let $\mu = 1$, $\lambda = 0$. For small $\ell > 0$ the operator A_{Σ_ℓ} has (at least) 2 eigenvalues, which satisfy

$$\Lambda - \lambda_1(\ell) = \ell^4 \cdot \nu_1 + \mathcal{O}(\ell^5) \qquad \text{as} \quad \ell \to 0,$$
$$\Lambda - \lambda_2(\ell) = \ell^8 \cdot \nu_2 + \mathcal{O}(\ell^9) \qquad \text{as} \quad \ell \to 0,$$

with $\nu_1, \nu_2 > 0$.

Principal characteristics of the problem:

- Non-additivity of the perturbation.
- Matrix structure of the differential operator.
- Empty discrete spectrum; indeed, $\sigma(A_{\Sigma_\ell}) = \sigma_{\mathrm{ess}}(A_{\Sigma_\ell}) = [0, \infty)$.

Ansatz:

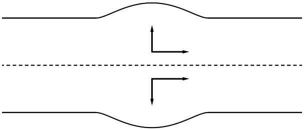
- Use the Dirichlet-to-Neumann mapping to transform the original problem into an boundary integral problem.

Internal symmetries

Reflection in the horizontal axis leads to a decomposition

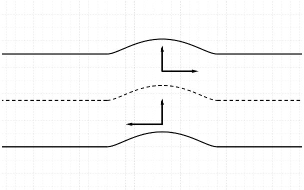
$$L_2(\Omega;\mathbb{C}^2) = H^{\mathrm{s}} \oplus H^{\mathrm{as}} \quad \text{and} \quad A_{\Sigma_\ell} = A^{\mathrm{s}}_{\Sigma_\ell} \oplus A^{\mathrm{as}}_{\Sigma_\ell}.$$

Symmetric waves



$$u_1(x_1,x_2) = u_1(x_1,-x_2)$$
$$u_2(x_1,x_2) = -u_2(x_1,-x_2).$$

Antisymmetric waves



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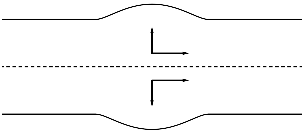

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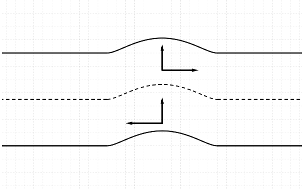

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Reduction to a mixed problem

Considering only symmetric waves we obtain a problem on the upper half-strip $\Omega_+ := \mathbb{R} \times \left(0, \frac{\pi}{2}\right)$. We search for $\lambda(\ell) \geq 0$ and $u \in H^1(\Omega_+; \mathbb{C}^2)$ such that

$$(-\Delta - \text{grad div})u = \lambda(\ell)u \qquad \text{in } \Omega_+$$

with boundary conditions

$$\begin{cases} (\partial_1 u_2 + \partial_2 u_1)\left(x_1, \frac{\pi}{2}\right) & = 0 \qquad \text{for } x_1 \in \mathbb{R}, \\ 2\partial_2 u_2 \left(x_1, \frac{\pi}{2}\right) & = 0 \qquad \text{for } x_1 \in \mathbb{R}, \end{cases}$$

$$\begin{cases} -(\partial_1 u_2 + \partial_2 u_1)(x_1, 0) & = 0 \qquad \text{for } x \in \mathbb{R}, \\ -2\partial_2 u_2(x_1, 0) & = 0 \qquad \text{for } x_1 \in \Sigma_\ell = (-\ell, \ell), \\ u_2(x_1, 0) & = 0 \qquad \text{for } x_1 \notin \Sigma_\ell. \end{cases}$$

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$$(-\Delta - \mathrm{grad}\ \mathrm{div})u = \cancel{\lambda(\ell)}\ \omega u \qquad \text{in } \Omega_+$$

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- Provide the boundary data g and calculate u.
- If $2\partial_2 u_2(x_1, 0) = 0$, $x_1 \in \Sigma_\ell$, then $\omega = \lambda(\ell)$.

The solution of the Poisson problem

Let $\omega \in \mathbb{C}\setminus\{0\}$. Applying the Fourier transform into the horizontal direction we obtain

$$\begin{pmatrix} 2\xi^2 - \partial_2^2 & -\mathrm{i}\xi\partial_2 \\ -\mathrm{i}\xi\partial_2 & \xi^2 - 2\partial_2^2 \end{pmatrix} \widehat{u}(\xi, x_2) = \omega \widehat{u}(\xi, x_2) \quad + \quad \text{(b.c.)}.$$

We have $\widehat{u}(\xi, x_2) = \sum_{i=1}^4 a_i(\xi,\omega) v_i(x_2)$, where

$$v_{1,2}(x_2) := \begin{pmatrix} \pm\beta \\ -\xi \end{pmatrix} e^{\pm \mathrm{i}\beta x_2}; \qquad v_{3,4}(x_2) := \begin{pmatrix} \xi \\ \pm\gamma \end{pmatrix} e^{\pm \mathrm{i}\gamma x_2};$$

with $\beta = \sqrt{\omega - \xi^2}$ and $\gamma = \sqrt{\frac{\omega}{2} - \xi^2}$. Inserting the boundary conditions leads to a linear system

$$L(\xi,\omega) a(\xi,\omega) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \widehat{g}(\xi) \end{pmatrix}, \qquad L(\xi,\omega) \in \mathbb{R}^{4\times 4}.$$

The Poisson and the Dirichlet-to-Neumann operator

We have

$$\det(L(\xi,\omega)) = 32\gamma^2(\gamma^2+\xi^2)\ \left[\sin\left(\beta\frac{\pi}{2}\right)\cos\gamma^3 + \cos\left(\beta\frac{\pi}{2}\right)\sin\left(\gamma\frac{\pi}{2}\right)\beta\xi^2\right].$$

Reminder: The Rayleigh-Lamb equation describes the spectrum of the unperturbed operator $A_\varnothing$.

We define for $\omega \notin [0,\infty) = \sigma(A_\varnothing+) = \sigma_{\mathrm{ess}}(A_{\Sigma_\ell+})$:

- Poisson operator $K_\omega : H^{1/2}(\mathbb{R}) \to H^1(\Omega_+;\mathbb{C}^2)$, $K_\omega g := u$.
- D-to-N operator $D_\omega : H^{1/2}(\mathbb{R}) \to H^{-1/2}(\mathbb{R})$, $D_\omega g := -2\partial_2 u_2|_{\mathbb{R}\times\{0\}}$.

We have $\widehat{D_\omega g}(\xi) = m_\omega(\xi)\widehat{g}(\xi)$ for some function m_ω:

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$$\frac{-2\sin\left(\frac{\beta\pi}{2}\right)\sin\left(\frac{\gamma\pi}{2}\right)[\gamma^6+2\gamma^2\xi^4+\xi^6]+4\left[\cos\left(\frac{\beta\pi}{2}\right)\cos\left(\frac{\gamma\pi}{2}\right)-1\right]\beta\gamma^3\xi^2}{(\gamma^2+\xi^2)[\sin\left(\frac{\beta\pi}{2}\right)\cos\left(\frac{\gamma\pi}{2}\right)\gamma^3+\cos\left(\frac{\beta\pi}{2}\right)\sin\left(\frac{\gamma\pi}{2}\right)\beta\xi^2]}$$

The truncated Dirichlet-to-Neumann operator

For $\omega \notin [0,\infty)$ the truncated Dirichlet-to-Neumann operator is given by

$$D_{\ell,\omega} : \mathrm{dom}(D_{\ell,\omega}) \to \mathrm{ran}(D_{\ell,\omega}), \qquad D_{\ell,\omega} := r_\ell D_\omega e_\ell,$$

where

- $e_\ell \cong$ extension by 0;
- $r_\ell \cong$ restriction to the interval $(-\ell,\ell)$;
- $\mathrm{dom}(D_{\ell,\omega}), \mathrm{ran}(D_{\ell,\omega})$ are suitable function spaces on $(-\ell,\ell)$.

The truncated Dirichlet-to-Neumann operator

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$$\boxed{D_{\ell,\omega} : H^{1/2}_{[-\ell,\ell]} \to H^{-1/2}(-\ell,\ell), \qquad D_{\ell,\omega} := r_\ell D_\omega e_\ell,}$$

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- $e_\ell \cong$ extension by 0;
- $r_\ell \cong$ restriction to the interval $(-\ell,\ell)$;

$$H^{1/2}_{[-\ell,\ell]} := \left\{ g \in L_2(-\ell,\ell) : e_\ell g \in H^{1/2}(\mathbb{R}) \right\},$$

$$H^{-1/2}(-\ell,\ell) := \left\{ h \in \mathscr{D}'(-\ell,\ell) : \exists \widetilde{h} \in H^{-1/2}(\mathbb{R}) \text{ s.t. } h = r_\ell \widetilde{h} \right\}.$$

Lemma

$$\omega \in \sigma_d(A_{\Sigma_\ell +}) \qquad \Longleftrightarrow \qquad \ker D_{\ell,\omega} \neq \{0\}.$$

Idea: Let $\ell \to 0$ and find $\omega = f(\ell)$ such that $\ker D_{\ell,\omega(\ell)} \neq \{0\}$.

⚠ We have $\sigma(A_{\Sigma_\ell +}) = [0,\infty)$, $\sigma_d(A_{\Sigma_\ell +}) = \varnothing$.

- Use the symmetry decomposition $L_2(\Omega_+;\mathbb{C}^2) = H_{1+} \oplus H_{2+}$ with $H_{1+} = \{(u_1(x_1),0)^T\}$. Then

$$A_{\Sigma +} := A^{(1)}_{\Sigma_\ell} \oplus A^{(2)}_{\Sigma_\ell}, \qquad \sigma_{\mathrm{ess}}(A^{(2)}_{\Sigma_\ell}) = [\Lambda,\infty)$$

with $\Lambda > 0$. Moreover,

$$K_\omega : H^{1/2}(\mathbb{R}) \to H^1(\Omega_+;\mathbb{C}^2) \cap H_{2+}, \qquad D_\omega : H^{1/2}(\mathbb{R}) \to H^{-1/2}(\mathbb{R})$$

are well-defined for $[\Lambda,\infty)$.

⚠ The domain of $D_{\ell,\omega}$ depends on ℓ.

- Let $T_\ell : L_2(-1,1) \to L_2(-\ell,\ell)$, $(T_\ell g)(x) = \ell^{-1/2} g(x/\ell)$ and define

$$Q(\ell,\omega) : H^{1/2}_{[-1,1]} \to H^{-1/2}(-1,1), \qquad Q(\ell,\omega) := T^*_\ell D_{\ell,\omega} T_\ell.$$

Analysis of the Dirichlet-to-Neumann operator

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A scaling argument

⚠ How to describe the D-to-N operator as $\ell \to 0$ and $\omega \to \Lambda$?

- Use the perturbation formula

$$D_\omega = D_0 - \omega K_0^* (I + \omega (A^{(2)}_{\varnothing +} - \omega)^{-1}) K_0$$

From $m_0(\xi) = |\xi| + O(1)$ we obtain

$$\begin{aligned}\langle Q(\ell,0)g,h\rangle &= \int_{\mathbb{R}} m_0(\xi/\ell)\widehat{g}(\xi)\,\overline{\widehat{h}(\xi)}\,\mathrm{d}\xi \\ &= \frac{1}{\ell}\int_{\mathbb{R}} |\xi|\cdot\widehat{g}(\xi)\,\overline{\widehat{h}(\xi)}\,\mathrm{d}\xi + \mathscr{O}(1) = \frac{1}{\ell}\langle Q_0 g,h\rangle + \mathscr{O}(1),\end{aligned}$$

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The unperturbed operator

Applying the Fourier transform in the horizontal direction one obtains a family of self-adjoint operators $(A_{\varnothing+})_{\xi\in\mathbb{R}}$, where

$$A_{\varnothing+}(\xi) := \begin{pmatrix} 2\xi^2 - \partial_2^2 & -\mathrm{i}\xi\partial_2 \\ -\mathrm{i}\xi\partial_2 & \xi^2 - 2\partial_2^2 \end{pmatrix},$$

$$D(A_{\varnothing+}(\xi)) := \{u \in H^2(I_+;\mathbb{C}^2) : \ \partial_2 u_2(\pm\pi/2) = 0 \wedge \\ \partial_2 u_1(\pm\pi/2) + \mathrm{i}\xi u_2(\pm\pi/2) = 0\}.$$

Then:

- $\omega \in \sigma(A_{\varnothing+}(\xi))$ if and only if
$$\sin\left(\beta\frac{\pi}{2}\right)\cos\left(\gamma\frac{\pi}{2}\right)\gamma^3 + \cos\left(\beta\frac{\pi}{2}\right)\sin\left(\gamma\frac{\pi}{2}\right)\beta\xi^2 = 0.$$
- If $\psi_\xi(x_2)$ is an eigenfunction of $A_{\varnothing+}(\xi)$ then $\psi_\xi(x_2)\cdot \mathrm{e}^{\mathrm{i}\xi x_1}$ is a generalised eigenfunction of $A_{\varnothing+}$.
- $\sigma(A_\varnothing+) = \cup_{\xi\in\mathbb{R}}\sigma(A_{\varnothing+}(\xi)) = [0,\infty)$.

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Applying the Fourier transform in the horizontal direction one obtains a family of self-adjoint operators $(A^{(2)}_{\varnothing +})_{\xi \in \mathbb{R}}$, where

$$A^{(2)}_{\varnothing +}(\xi) := \begin{pmatrix} 2\xi^2 - \partial_2^2 & -\mathrm{i}\xi\partial_2 \\ -\mathrm{i}\xi\partial_2 & \xi^2 - 2\partial_2^2 \end{pmatrix},$$

$$D(A^{(2)}_{\varnothing +}(\xi)) := \{u \in H^2(I_+;\mathbb{C}^2) : \ \partial_2 u_2(\pm\pi/2) = 0 \wedge u \perp 1 \wedge \\ \partial_2 u_1(\pm\pi/2) + \mathrm{i}\xi u_2(\pm\pi/2) = 0\}.$$

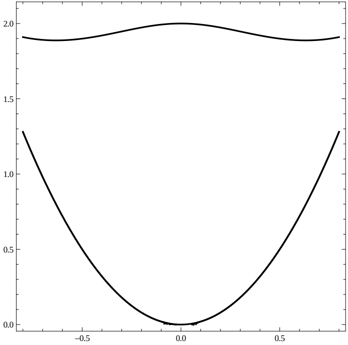
Then:

- $\omega \in \sigma(A^{(2)}_{\varnothing +}(\xi))$ if and only if $\omega \neq 2\xi^2$ and

$$\sin\left(\beta\frac{\pi}{2}\right)\cos\left(\gamma\frac{\pi}{2}\right)\gamma^3 + \cos\left(\beta\frac{\pi}{2}\right)\sin\left(\gamma\frac{\pi}{2}\right)\beta\xi^2 = 0.$$

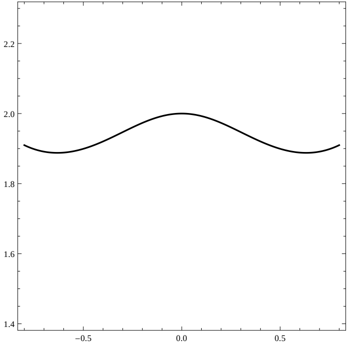
- If $\psi_\xi(x_2)$ is an eigenfunction of $A^{(2)}_{\varnothing +}(\xi)$ then $\psi_\xi(x_2)\cdot \mathrm{e}^{\mathrm{i}\xi x_1}$ is a generalised eigenfunction of $A^{(2)}_{\varnothing +}$.
- $\sigma(A^{(2)}_{\varnothing +}) = \cup_{\xi\in\mathbb{R}}\sigma(A^{(2)}_{\varnothing +}(\xi)) = [\Lambda, \infty)$:

The dispersion curves of $A_{\varnothing +}$



- x-axis: parameter ξ.
- y-axis: eigenvalues of $A_{\varnothing +}(\xi)$.

The dispersion curves of $A^{(2)}_{\varnothing +}$



- x-axis: parameter ξ.
- y-axis: eigenvalues of $A^{(2)}_{\varnothing +}(\xi)$.

Let $\zeta_1(\xi)$ be the lowest eigenvalue branch of $A^{(2)}_{\varnothing +}$. Then

$$\Lambda = \min\{\zeta_1(\xi) : \xi \in \mathbb{R}\} = \inf \sigma(A^{(2)}_{\varnothing +}) = \inf \sigma_{\mathrm{ess}}(A^{(2)}_{\Sigma_\ell +}).$$

We have $\zeta_1(\pm\kappa) = \Lambda$,

$$\kappa = 0.632138 \pm 10^{-6} \qquad \text{and} \qquad \Lambda = 1.887837 \pm 10^{-6}.$$

Estimate of the resolvent term

Lemma

We have

$$\omega T_\ell^* r_\ell K_0^* (I + \omega(A^{(2)}_{\varnothing +} - \omega)^{-1}) K_0 r_\ell T_\ell$$
$$= \frac{8 \cdot |\partial_2 \psi_{\kappa,2}(0)|^2}{\sqrt{\Lambda - \omega} \cdot \sqrt{2\zeta_1''(\kappa)}} T_\ell^* (P_+ + P_-) T_\ell + \mathcal{O}(1).$$

The remainder may be estimated uniformly in the operator norm of $L_2(-1,1)$.

Here

- $P_\pm$ is the projection in $L_2(-1,1)$ onto the subspace spanned by $\Phi_\pm(x_1) := \mathrm{e}^{\pm \mathrm{i}\kappa x_1}$
- $\psi_{\pm\kappa} \in L_2(I_+; \mathbb{C}^2)$ is chosen such that

$$A^{(2)}_{\varnothing +}(\pm\kappa)\psi_{\pm\kappa} = \Lambda \psi_{\pm\kappa} \qquad \text{and} \qquad \|\psi_{\pm\kappa}\|_{L_2(I_+;\mathbb{C}^2)} = 1, \qquad (1)$$

Idea of the proof

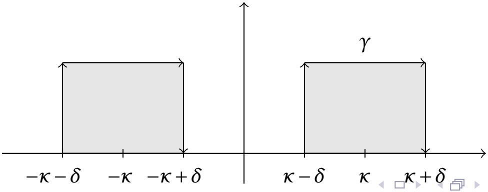
We use a resolvent expansion of $A^{(2)}_{\varnothing +}$ near the bottom of the essential spectrum. For $f, g \in L_2(\Sigma_\ell)$ we have

$$\langle K_0^*(A^{(2)}_{\varnothing +} - \omega)^{-1} K_0 f, g\rangle = \int_{\mathbb{R}} \langle K_0(\xi)(A^{(2)}_{\varnothing +}(\xi) - \omega)^{-1} \widehat{g}(\xi), K_0(\xi)\widehat{h}(\xi)\rangle \, \mathrm{d}\xi$$

where $K_0(\cdot)$ is the paremeter-dependent Poisson operator. From the spectral theorem we obtain

$$(A^{(2)}_{\varnothing +}(\xi) - \omega)^{-1} = \sum_{k=1}^{\infty} \frac{1}{\zeta_k(\xi) - \omega} P_k(\xi) = \frac{1}{\zeta_1(\xi) - \omega} + \mathscr{O}(1).$$

Finally, we use $\zeta_1(\xi) \sim \Lambda + \zeta_1''(\pm\kappa)\xi^2$ near $\pm\kappa$ and change the path of integration



The asymptotic formula

We obtain

$$\ell \cdot Q(\ell,\omega) = Q(\ell,0) - \omega T_\ell^* r_\ell K_0^* (I + \omega (A_{\varnothing+}^{(2)} - \omega)^{-1}) K_0 r_\ell T_\ell$$

$$\sim Q_0 - \frac{8\ell \cdot |\partial_2 \psi_{\pm\kappa,2}(0)|^2}{\sqrt{\Lambda - \omega} \cdot \sqrt{2\zeta_1''(\kappa)}} \, T_\ell^* (P_+ + P_-) T_\ell,$$

Lemma (Birman-Schwinger principle)

Let $T : D(T) \subseteq H \to H$ be invertible and let $V \in \mathscr{L}(H)$, $V \geq 0$ be a rank-one perturbation. For $\alpha > 0$ we have

$$\ker(T - \alpha V) \neq \{0\} \quad \Longleftrightarrow \quad \alpha \cdot \mathrm{tr}(V^{1/2} T^{-1} V^{1/2}) = 1.$$

Apply the B-S principle with $V = V(\ell) := T_\ell^* (P_+ + P_-) T_\ell$.

⚠ $T_\ell^* (P_+ + P_-) T_\ell$ is a rank-two perturbation.

Solution: Use an additional symmetry decomposition:

$$L_2(-1,1) = L_{2,\mathrm{even}}(-1,1) \oplus L_{2,\mathrm{odd}}(-1,1).$$

If $f,g \in L_2(-1,1)$ are even with respect to $x_1 = 0$, then

$$\begin{aligned}
&\langle T_\ell^*(P_+ + P_-)T_\ell f, g\rangle \\
&= \ell \left(\int_{(-1,1)} f(x_1)\cos(\kappa x_1 \ell)\,\mathrm{d}x_1\right) \cdot \left(\int_{(-1,1)} \cos(\kappa x_1 \ell)\overline{g(x_1)}\,\mathrm{d}x_1\right) \\
&= \ell \langle f, \mathbf{1}\rangle \cdot \langle \mathbf{1}, g\rangle + \mathcal{O}(\ell^2).
\end{aligned}$$

If f,g are odd, then

$$\begin{aligned}
&\langle T_\ell^*(P_+ + P_-)T_\ell f, g\rangle \\
&= \ell \left(\int_{(-1,1)} f(x_1)\sin(\kappa x_1 \ell)\,\mathrm{d}x_1\right) \cdot \left(\int_{(-1,1)} \sin(\kappa x_1 \ell)\overline{g(x_1)}\,\mathrm{d}x_1\right) \\
&= \ell^3 \kappa^2 \langle f, \mathbf{x_1}\rangle \cdot \langle \mathbf{x_1}, g\rangle + \mathcal{O}(\ell^4).
\end{aligned}$$

Recall that $P_\pm$ are the projections in $x_1 \mapsto \mathrm{e}^{\pm \mathrm{i}\kappa x_1}$.

The asymptotic formula

Thus, in $L_{2,s}(-1,1)$ we obtain

$$\ell Q_{\ell,\omega} \sim Q_0 - \frac{8\ell^2 \cdot |\partial_2 \psi_{\pm\kappa,2}(0)|^2}{\sqrt{\Lambda - \omega} \cdot \sqrt{2\zeta_1''(\kappa)}} V_{\mathbf{1}},$$

where $\langle V_{\mathbf{1}} f, g\rangle = \langle f, \mathbf{1}\rangle \cdot \langle \mathbf{1}, g\rangle$. Then

$$\begin{aligned}
\omega \text{ eigenvalue of } A_\ell \quad &\Longleftrightarrow \quad 0 \in \ker D_{\ell,\omega} \\
&\Longleftrightarrow \quad \frac{8\ell^2 \cdot |\partial_2 \psi_{\pm\kappa,2}(0)|^2}{\sqrt{\Lambda - \omega} \cdot \sqrt{2\zeta_1''(\kappa)}} \operatorname{tr}(V_{\mathbf{1}}^{1/2} Q_0^{-1} V_{\mathbf{1}}^{1/2}) \sim 1 \\
&\Longleftrightarrow \quad \sqrt{\Lambda - \omega} \sim \frac{8\ell^2 \cdot |\partial_2 \psi_{\pm\kappa,2}(0)|^2}{\sqrt{2\zeta_1''(\kappa)}} \operatorname{tr}(V_{\mathbf{1}}^{1/2} Q_0^{-1} V_{\mathbf{1}}^{1/2}).
\end{aligned}$$

Thus, in $L_{2,s}(-1,1)$ we obtain

$$\ell Q_{\ell,\omega} \sim Q_0 - \frac{8\ell^2 \cdot |\partial_2 \psi_{\pm\kappa,2}(0)|^2}{\sqrt{\Lambda-\omega}\cdot\sqrt{2\zeta_1''(\kappa)}} V_{\mathbf{1}},$$

where $\langle V_{\mathbf{1}} f, g\rangle = \langle f, \mathbf{1}\rangle \cdot \langle \mathbf{1}, g\rangle$. Then

$$\begin{aligned} \lambda(\ell) \text{ eigenvalue of } A_\ell \quad &\Longleftrightarrow \quad 0 \in \ker D_{\ell,\omega} \\ &\Longleftrightarrow \quad \frac{8\ell^2 \cdot |\partial_2 \psi_{\pm\kappa,2}(0)|^2}{\sqrt{\Lambda-\lambda(\ell)}\cdot\sqrt{2\zeta_1''(\kappa)}} \mathrm{tr}(V_{\mathbf{1}}^{1/2} Q_0^{-1} V_{\mathbf{1}}^{1/2}) \sim 1 \\ &\Longleftrightarrow \quad \sqrt{\Lambda-\lambda(\ell)} \sim \frac{8\ell^2 \cdot |\partial_2 \psi_{\pm\kappa,2}(0)|^2}{\sqrt{2\zeta_1''(\kappa)}} \mathrm{tr}(V_{\mathbf{1}}^{1/2} Q_0^{-1} V_{\mathbf{1}}^{1/2}). \end{aligned}$$

Main results

Theorem (2D, $\Sigma_\ell := (-\ell, \ell)$)

For small $\ell > 0$ there exists exactly two eigenvalues, which satisfy

$$\Lambda - \lambda_1(\ell) = \ell^4 \cdot \frac{16\pi^2 |\partial_2 \psi_2(0)|^4}{\zeta_1''(\kappa)} + \mathcal{O}(\ell^5) \qquad \text{as} \quad \ell \to 0,$$

$$\Lambda - \lambda_2(\ell) = \ell^8 \cdot \frac{\pi^2 \kappa^4 |\partial_2 \psi_2(0)|^4}{4\zeta_1''(\kappa)} + \mathcal{O}(\ell^9) \qquad \text{as} \quad \ell \to 0.$$

Theorem (3D, $\Sigma_\ell := B(0, \ell)$)

For every $m \in \mathbb{Z}$ there exists an eigenvalue $\lambda(\ell, m)$ such that

$$\Lambda - \lambda(\ell, m) = \ell^{6+4|m|} \cdot \frac{16\kappa^{4|m|+2} \cdot |\partial_3 \psi_3(0)|^4}{2^{4|m|} \cdot f''(\kappa)} \cdot \rho_m + \mathcal{O}(\ell^{7+4|m|}),$$

as $\ell \to 0$ for some constant $\rho_m > 0$.

Uniqueness of the eigenvalue and the 3D case

Note that the Dirichlet-to-Neumann operator acting on $L_{2,\mathrm{s}}(-1,1)$ has a complete system of eigenfunctions corresponding to eigenvalues

$$\mu_1(\ell,\omega) \le \mu_2(\ell,\omega) \le \dots$$

One can show:

- $\mu_2(\ell,\omega) > 0$ for small $\ell > 0$.
- $\mu_1(\ell,\omega)$ strictly decreasing in ω.
- $\mu_1(\ell,\omega)$ strictly increasing in ℓ.

A similar assertion holds true for the part acting $L_{2,\mathrm{as}}(-1,1)$.

In 3D we use the symmetry decomposition for the L_2 space on the crack $L_2(B(0,1)) = \oplus_{m\in\mathbb{Z}} L_{2,m}(B(0,1))$, where

$$L_{2,m}(B(0,1)) := \{g \in L_2(B(0,1)) : g(r\cos\varphi, r\sin\varphi) = \mathrm{e}^{\mathrm{i}m\varphi}\widetilde{g}(r)$$
$$\text{for some } \widetilde{g} : (0,1) \to \mathbb{C}\}.$$

Thank you for your attention!