The Calderón problem in transversally anisotropic geometries

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Outline

The anisotropic Calderón problem

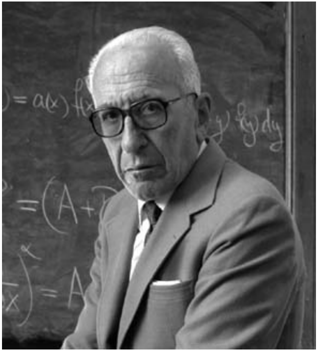
The Dirichlet-to-Neumann map in waveguides

Wick rotation

The Calderón problem and the X-ray transform

The Calderón problem

On an inverse boundary value problem,
Seminar on Numerical Analysis and its Applications to Continuum Physics, Rio de Janeiro,
Editors W.H. Meyer and M.A. Raupp,
Sociedade Brasileira de Matematica (1980), 65–73.



http://en.wikipedia.org/wiki/Alberto_Calderón

In a foundational paper of 1980, A. Calderón asked the following question:
Is it possible to determine the electrical conductivity of a body by making current and voltage measurements at the boundary?

Riemannian rigidity

In fact, one can state the inverse problem with a geometric flavour.

Let (M, g) be a compact Riemannian manifold with boundary ∂M of dimension $n \geq 3$ and q a bounded measurable function. Consider the Dirichlet problem

$$\begin{cases} (-\Delta_g + q)u = 0 \\ u|_{\partial M} = f \in H^{\frac{1}{2}}(\partial M) \end{cases}$$

and define the associated Dirichlet-to-Neumann map under the assumption that 0 is not a Dirichlet eigenvalue of $-\Delta_g + q$

$$\Lambda_{g,q} u = \partial_\nu u|_{\partial M}$$

where ν is a unit normal to the boundary.
If $q = 0$, we use $\Lambda_g = \Lambda_{g,0}$ as a short notation.

Riemannian rigidity

The inverse problem is whether the DN map determines the metric g.
There is a gauge invariance, that is by isometries which leave the boundary points unchanged:

$$\Lambda_{\varphi^* g} = \Lambda_g, \quad \varphi|_{\partial M} = \mathrm{Id}_{\partial M}$$

Inverse problem: Does the Dirichlet-to-Neumann map $\Lambda_{g,q}$ determine the potential q and the metric g modulo such isometries?

If $n \geq 3$ and $q = 0$ this is a generalization of the anisotropic conductivity problem and one passes from one to the other by

$$\gamma^{jk} = \sqrt{\det g}\, g^{jk}, \quad g^{jk} = (\det \gamma)^{-\frac{2}{n-2}} g^{jk}.$$

Conformal metrics

There is a conformal gauge transformation

$$\Delta_{cg} u = c^{-1}(\Delta_g + q_c)(c^{\frac{n-2}{4}} u), \quad q_c = c^{\frac{n+2}{4}} \Delta_g (c^{\frac{n-2}{4}})$$

which translates at the boundary into

$$\Lambda_{cg,q} f = c^{-\frac{n+2}{4}} \Lambda_{g,q+q_c}(c^{\frac{n-2}{4}} u) + \frac{n-2}{4} c^{-\frac{1}{2}} \partial_\nu c f.$$

So if one knows $c, \partial_\nu c$ at the boundary (boundary determination) then one can deduce one DN map from the other.

A more reasonable inverse problem: $\Lambda_{cg} = \Lambda_g \Rightarrow c = 1$.
Note that there is no isometry gauge invariance in this case.

Some references $n \geq 3$

1987 Sylvester-Uhlmann: isotropic case

1989 Lee-Uhlmann: boundary determination, analytic metrics, no potential, determination of the metric

2001 Lassas-Uhlmann: improvement on topological assumptions

2007 Kenig-Sjöstrand-Uhlmann: small subsets of the boundary, $n \geq 3$, global Carleman estimates with logarithmic weights, introduction of limiting Carleman weights.

2009 Guillarmou-Sa Baretto: Einstein manifolds, no potential, determination of the metric, unique continuation argument

2009 DSF-Kenig-Salo-Uhlmann: fixed admissible geometries, determination of a smooth potential, CGOs

2011 DSF-Kenig-Salo: fixed admissible geometries, determination of an unbounded potential, CGOs

Remarks

1. Analytic metrics case fairly well understood. The smooth case remains a challenging problem.
2. There are limitations in the method using CGO construction: the existence of limiting Carleman weights
3. Identifiability of the metric within a conformal class

$$\Lambda_{cg} = \Lambda_g \Rightarrow c = 1.$$

4. With boundary determination

$$\Lambda_{cg} = \Lambda_g \Rightarrow c|_{\partial M} = 1, \quad \partial_\nu c|_{\partial M} = 0.$$

it is enough to solve the inverse problem on the Schrödinger equation with a fixed metric

$$\Lambda_{g,q_1} = \Lambda_{g,q_2} \Rightarrow q_1 = q_2.$$

Transversally anisotropic geometries

Geometrical setting : consider a cylinder $T = \mathbf{R} \times M_0$ where (M_0, g_0) is a compact Riemannian manifold with boundary.

Endow T with the following Riemannian metric

$$g = \mathrm{d}t^2 + g_0$$

t will always denote the Euclidean variable. and x the variables in M_0.
Consider the following Schrödinger operator

$$-\Delta_g + q = -\partial_t^2 - \Delta_{g_0} + q$$

where $q \in L^\infty(T)$ is real valued. Note that the transversal metric g_0 does not depend on t.

Time independent potentials

We are interested in two different situations :

1. we restrict the setting to a compact Riemannian manifold with boundary M which embeds (conformally) in T, and consider the inverse problem of recovering the potential from the Dirichlet-to-Neumann map.
2. we consider the non-compact case and suppose that the potential q does not depend on t.

Wave guides

- Situation 1 corresponds to trying to solve the anisotropic Calderón problem in a conformal class. The conformal transversally anisotropic geometry is justified by the requirement of having limiting Carleman weights, and the ability to construct complex geometrical optics solutions with antagonising exponential behaviours to the Schrödinger equation.
- Situation 2 corresponds to the structure of wave guides. The independence of both the transversal metric g_0 and the potential q with respect to t will allow to fully recover the metric (up to isometries) and the potential.

In this talk, I will mainly be concerned with the second situation.

Outline

The anisotropic Calderón problem

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Spectral description

Geometrical setting : $T = \mathbf{R} \times M_0$, $g = \mathrm{d}t^2 + g_0$, $\Delta_g = \partial_t^2 + \Delta_{g_0}$, potential $q_0 : M_0 \to \mathbf{R}$ independent on t.
Denote by $\lambda_1 \leq \lambda_2 \leq \cdots$ the Dirichlet eigenvalues of $-\Delta_{g_0} + q_0$ on M_0 and consider

$$L^2_\delta(T) = \left\{ f \in L^2_{\mathrm{loc}}(T) : \langle t \rangle^\delta f \in L^2(T) \right\}$$

$\tilde{u}(t, l)$ denotes the Fourier coefficient of u.

- $-\Delta_g + q_0$ with domain $H^1_0(T) \cap H^2(T)$ is self-adjoint on $L^2(T)$
- the spectrum of $-\Delta_g + q_0$ is $[\lambda_1, \infty)$
- if $\lambda \in \mathbf{C} \setminus [\lambda_1, \infty)$ then for any $\delta \in \mathbf{R}$, $(-\Delta_g + q_0 - \lambda)^{-1} : L^2_\delta(T) \to \left\{ u \in H^2_\delta(T) : u|_{\partial T} = 0 \right\}$
- if $\lambda \in [\lambda_1, \infty)$ but $\lambda \neq \lambda_l$ then for any $\delta > 1/2$, $(-\Delta_g + q_0 - \lambda - i0)^{-1} : L^2_\delta(T) \to \left\{ u \in H^2_{-\delta}(T) : u|_{\partial T} = 0 \right\}$

The DN map at $\lambda \in \mathbf{C} \setminus [\lambda_1, \infty)$

If $\lambda \in \mathbf{C} \setminus [\lambda_1, \infty)$, then for any $f \in H^{3/2}(\partial T)$ there is a unique solution $u \in H^2(T)$ of the equation

$$(-\Delta + q_0 - \lambda)u = 0 \text{ in } T, \qquad u|_{\partial T} = f.$$

If $f \in C_c^\infty(\partial T)$ then $u \in C^\infty(T)$ and there is a linear map

$$\begin{aligned} \Lambda_{g_0,q_0}^{Ell}(\lambda) : C_c^\infty(\partial T) &\to C^\infty(\partial T), \\ f &\mapsto \partial_\nu u|_{\partial T}. \end{aligned}$$

For any $\delta \in \mathbf{R}$, this map extends as a bounded linear map

$$\Lambda_{g_0,q_0}^{Ell}(\lambda) : H_\delta^{3/2}(\partial T) \to H_\delta^{1/2}(\partial T).$$

Obstruction to uniqueness when $\lambda \in [\lambda_1, \infty), \lambda \neq \lambda_l$

Choose $l_0 \geq 1$ so that $\lambda_{l_0} < \lambda < \lambda_{l_0+1}$. Note that

$$\sum_{l=1}^{l_0} c_l^{\pm} \mathrm{e}^{\pm i\sqrt{\lambda-\lambda_l}t} \varphi_l(x)$$

(recall that (φ_l) are Dirichlet eigenfunctions) is a solution to

$$(-\Delta_g + q_0)u = 0, \quad u|_{\partial T} = 0$$

and belongs to $L^2_\delta(\partial T)$ with $\delta > 1/2$. One has to impose conditions to guarantee uniqueness and have a well-behaved DN map.

The DN map at $\lambda \in [\lambda_1, \infty), \lambda \neq \lambda_l$

Choose $l_0 \geq 1$ so that $\lambda_{l_0} < \lambda < \lambda_{l_0+1}$. Let $\delta > 1/2$, and let $m \geq 2$, for any $H_\delta^{m-1/2}(\partial T)$, the equation

$$(-\Delta + q_0 - \lambda)u = 0 \text{ in } T, \qquad u|_{\partial T} = f$$

has a unique solution $u \in H^m_{-\delta}(T)$ satisfying the outgoing radiation condition

$$(\partial_t \mp i\sqrt{\lambda - \lambda_l})\tilde{u}(t,l) \to 0 \quad \text{as } t \to \pm\infty \text{ for all } 1 \leq l \leq l_0.$$

If $f \in C_c^\infty(\partial T)$, then $u \in C^\infty(T)$ and there is a linear map

$$\Lambda^{Ell}_{g_0,q_0}(\lambda) : C_c^\infty(\partial T) \to C^\infty(\partial T), \quad f \mapsto \partial_\nu u|_{\partial T}.$$

For any $\delta > 1/2$, this map extends as a bounded linear map

$$\Lambda^{Ell}_{g_0,q_0}(\lambda) : H_\delta^{m-1/2}(\partial T) \to H_{-\delta}^{m-3/2}(\partial T).$$

Main results

Theorem

Let (M_0, g_0) and $(M_0, \tilde{g}_0)$ be two compact manifolds with boundary ∂M_0, and let $q_0, \tilde{q}_0 \in C^\infty(M_0)$. If

$$\Lambda^{Ell}_{g_0,q_0}(\lambda) = \Lambda^{Ell}_{\tilde{g}_0,\tilde{q}_0}(\lambda) \quad \textit{for some } \lambda \in \mathbf{C} \setminus ([\lambda_1, \infty) \cup [\tilde{\lambda}_1, \infty)),$$

then $\tilde{g}_0 = \psi_0^* g_0$ for some diffeomorphism $\psi_0 : M_0 \to M_0$ with $\psi_0|_{\partial M_0} =$ Id, and also $\tilde{q}_0 = q_0$.

Theorem

Given the data $(\partial T, \Lambda^T_{g_0,q_0}(\lambda))$ for a fixed $\lambda \in [\lambda_1, \infty) \setminus \{\lambda_1, \lambda_2, \ldots\}$, where $\partial T = \mathbf{R} \times \partial M_0$ and $\Lambda^{Ell}_{g_0,q_0}(\lambda) : C_c^\infty(\partial T) \to C^\infty(\partial T)$ corresponds to the Schrödinger operator $-\Delta + q_0$ on T, one can reconstruct the potential q_0 and a Riemannian manifold $(\hat{M}_0, \hat{g}_0)$ isometric to (M_0, g_0).

Outline

The Gel'fand inverse problem

Consider the wave equation

$$\begin{aligned}
(\partial_t^2 - \Delta_{g_0} + q_0)u = 0 \quad &\text{in } (0,T) \times M_0, \\
u(0) = \partial_t u(0) = 0&, \\
u|_{(0,T)\times\partial M_0} = f&.
\end{aligned}$$

This problem has a unique solution $u \in C^\infty((0,T) \times M_0)$ for any $f \in C_c^\infty((0,T) \times \partial M_0)$, and we can define the hyperbolic DN map

$$\begin{aligned}
\Lambda_{g_0,q_0}^{Hyp} : C_c^\infty((0,T) \times \partial M_0) &\to C^\infty((0,T) \times \partial M_0), \\
f &\mapsto \partial_\nu u|_{(0,T)\times\partial M_0}.
\end{aligned}$$

The inverse problem is to determine the metric g_0 up to isometry and the potential q_0 from the knowledge of the DN map Λ_{g_0,q_0}^{Hyp}.

The boundary control method by Belishev and Kurylev

The Gel'fand problem has a positive answer under the condition that $T > 2r(M_0)$ where $r(M_0)$ is the time needed to fill in the manifold by waves from the boundary. This follows from the boundary control method introduced by Belishev. The boundary control method is based on three components:

1. Integration by parts (Blagovestchenskii identity): recover inner products of solutions at a fixed time from the hyperbolic DN map.
2. Approximate controllability based on an unique continuation theorem of Tataru: solutions $u(t_0, \cdot)$ are L^2 dense in the appropriate domain of influence.
3. Recovering the coefficients: this uses a boundary distance representation of (M_0, g_0) together with projectors to domains of influence and special solutions such as Gaussian beams.

“Time independent” inverse problems

It was proved by Katchalov, Kurylev, Lassas and Mandache that knowing the transversal DN maps $\{\Lambda^{Tr}_{g_0,q_0}(\mu)\}_{\mu\in\mathbf{C}}$

$$\Lambda^{Tr}_{g_0,q_0}(\lambda) : v|_{\partial M_0} \mapsto \partial_\nu v|_{\partial M_0}, \quad (-\Delta_{g_0} + q_0 - \lambda)v = 0 \text{ in } M_0.$$

is equivalent to knowing the DN map for the following equations:

- Wave equation $(\partial_t^2 - \Delta_{g_0} + q_0)u = 0$ in $(0,\infty) \times M_0$,
- Heat equation $(\partial_t - \Delta_{g_0} + q_0)u = 0$ in $(0,\infty) \times M_0$,
- Schrödinger equation $(i\partial_t - \Delta_{g_0} + q_0)u = 0$ in $(0,\infty) \times M_0$.

Our results show that the elliptic equation $(-\partial_t^2 - \Delta_{g_0} + q_0)u = 0$ in $\mathbf{R} \times M_0$ can be added to this list.

Some references

1987 Belishev boundary control method

1992 Belishev and Kurylev, determination of the metric from the hyperbolic DN map

1995 Tataru, unique continuation for the wave equation with time independent coefficcients

2001 Katchalov, Kurylev and Lassas, book Inverse boundary spectral probles

2004 Katchalov, Kurylev, Lassas and Mandache, Time independent inverse problems

Summary of the argument

The argument proceeds roughly as follows:

1. Extend $\Lambda^{Ell}_{g_0,q_0}(\lambda)$ to act on weighted Sobolev spaces on ∂T.
2. If $k>0$, obtain $\Lambda^{Tr}_{g_0,q_0}(\lambda-k^2)$ for any $h\in C^\infty(\partial M_0)$ via

$$\Lambda^{Tr}_{g_0,q_0}(\lambda-k^2)h=\mathrm{e}^{-ikt}\Lambda^{Ell}_{g_0,q_0}(\lambda)(\mathrm{e}^{ikt}h).$$

3. Recover $\Lambda^{Tr}_{g_0,q_0}(\mu)$ for $\mu\in\mathbf{C}$ from $\{\Lambda^{Tr}_{g_0,q_0}(\lambda-k^2)\}_{k>0}$ by meromorphic continuation.
4. Recover $\Lambda^{Hyp}_{g_0,q_0}$ from $\{\Lambda^{Tr}_{g_0,q_0}(\mu)\}_{\mu\in\mathbf{C}}$ by Laplace transform in time.
5. Use the boundary control method to determine (M_0,g_0) up to isometry and q_0 from $\Lambda^{Hyp}_{g_0,q_0}$.

From the elliptic to the hyperbolic DN

Case $\lambda \notin [\lambda_1, \infty)$

If $k \in \mathbf{R}$ then for all $h \in H^{3/2}(\partial M_0)$

$$\Lambda^{Tr}_{g_0,q_0}(\lambda - k^2)h = \mathrm{e}^{-ikt}\Lambda^{Ell}_{g_0,q_0}(\lambda)(\mathrm{e}^{ikt}h|_{\partial T}).$$

This implies that the right is independent of the t variable.

Since $\lambda - k^2$ is not a Dirichlet eigenvalue of $-\Delta_{g_0} + q_0$, one can find $v_h \in H^2(M_0)$ solving

$$(-\Delta + q_0 - (\lambda - k^2))v_h = 0 \text{ in } M_0, \qquad v_h|_{\partial M_0} = h.$$

and define $f(t,x) = \mathrm{e}^{ikt}h(x)$.

Since k is real, we have $f \in H^{3/2}_\delta(\partial T)$ for any $\delta < -1/2$, and the function $u(t,x) = \mathrm{e}^{ikt}v_h(x)$ is in $H^2_\delta(T)$ and solves

$$(-\partial_t^2 - \Delta_{g_0} + q_0 - \lambda)u = 0 \text{ in } T, \qquad u|_{\partial T} = f.$$

Thus

$$\Lambda^{Ell}_{g_0,q_0}(\lambda)f = \partial_\nu u|_{\partial T} = \mathrm{e}^{ikt}(\partial_\nu v_h|_{\partial M_0}) = \mathrm{e}^{ikt}\Lambda^{Tr}_{g_0,q_0}(\lambda - k^2)h.$$

From the elliptic to the hyperbolic DN

Case $\lambda \in [\lambda_1, \infty)$, $\lambda \neq \lambda_l$

Recall that when $\lambda \in \mathbf{C} \setminus [\lambda_1, \infty)$, the main point was

$$\mathrm{e}^{ikt} \Lambda_{g_0,q_0}^{M_0}(\lambda - k^2) h = \Lambda_{g_0,q_0}^{T}(\lambda)(e^{ikt} h|_{\partial T}), \quad h \in H^{3/2}(\partial M_0).$$

This identity does not directly generalize to the case where λ is in the continuous spectrum, because the boundary value $e^{ikt} h|_{\partial T}$ on the right hand side is not in $H_\delta^{3/2}(\partial T)$ for $\delta > 1/2$.

However, by using suitable cutoff and averaging arguments we can still recover the transversal DN maps from $\Lambda_{g_0,q_0}^{T}(\lambda)$.

$$\mathrm{e}^{ikt} \Lambda_{g_0,q_0}^{Tr}(\lambda - k^2) h = \lim_{R \to \infty} \frac{1}{R-1} \int_1^R \Lambda_{g_0,q_0}^{Ell}(\lambda)(\mathrm{e}^{ikt} \Psi_{R'}(t) h|_{\partial T})\, dR'$$

where $\Psi_R \in C_c^\infty(\mathbf{R})$ is a suitable cutoff function.

From the elliptic to the hyperbolic DN

Case $\lambda \in [\lambda_1, \infty)$, $\lambda \neq \lambda_l$

Take $k \in \mathbf{R}$ such that $\lambda - k^2 \neq \lambda_l$ then

$$\mathrm{e}^{ikt}\Lambda^{Tr}_{g_0,q_0}(\lambda - k^2)h = \lim_{R\to\infty}\frac{1}{R-1}\int_1^R \Lambda^{Ell}_{g_0,q_0}(\lambda)(\mathrm{e}^{ikt}\Psi_{R'}(t)h|_{\partial T})\,dR'$$

Prove the identity on the low and high frequency projections $h_1 + h_2 = h$.

Low frequency part of the solution:

$$u_{1,'}(t,x) = \frac{1}{2i}\sum_{l=1}^{l_0}(\lambda-\lambda_l)^{-1/2}\left(\int_{-\infty}^{\infty}\mathrm{e}^{i|t-t'|\sqrt{\lambda-\lambda_l}}\tilde{\eta}_{R'}(t',l)\,\mathrm{d}t'\right)\varphi_l(x)$$

$$\tilde{\eta}_{R'}(t',l) = \mathrm{e}^{ikt'}\Psi_{R'}(t')\int_{\partial M_0}h_1(y)\partial_\nu\varphi_l(y)\,\mathrm{d}S(y)$$

$$\partial_\nu u_{1,R'}(t,x) = \frac{1}{2i}\sum_{l=1}^{l_0}(\lambda-\lambda_l)^{-1/2}\left(\int_{-\infty}^{\infty}\mathrm{e}^{i|t-t'|\sqrt{\lambda-\lambda_l}}\tilde{\eta}(t',l)\,\mathrm{d}t'\right)\partial_\nu\varphi_l(x)$$

From the elliptic to the hyperbolic DN

Case $\lambda \in [\lambda_1, \infty)$, $\lambda \neq \lambda_l$

Replacing $\Psi_{R'}$ by $1_{[-R',R']}$, the t' integral can be computed explicitly :

$$\int_{-R'}^{R'} \mathrm{e}^{i|t-t'|\sqrt{\lambda-\lambda_l}} \mathrm{e}^{ikt'}\, dt' = \frac{2i(\lambda-\lambda_l)^{1/2}}{\lambda-\lambda_l-k^2}\mathrm{e}^{ikt} + \frac{\mathrm{e}^{i(k+\sqrt{\lambda-\lambda_l})R'-it\sqrt{\lambda-\lambda_l}}}{i(k+\sqrt{\lambda-\lambda_l})} - \frac{\mathrm{e}^{-i(k-\sqrt{\lambda-\lambda_l})R'+it\sqrt{\lambda-\lambda_l}}}{i(k-\sqrt{\lambda-\lambda_l})}.$$

The last two terms oscillate with respect to R'; we can remove these oscillating terms by averaging:

$$\lim_{R\to\infty} \frac{1}{R-1}\int_1^R \mathrm{e}^{i(k\pm\sqrt{\lambda-\lambda_l})R'}\, dR' = 0.$$

since $k \pm \sqrt{\lambda-\lambda_l} \neq 0$.

From the elliptic to the hyperbolic DN

Case $\lambda \in [\lambda_1, \infty)$, $\lambda \neq \lambda_l$

This shows that for any fixed $(t, x) \in \partial T$, we have

$$\lim_{R\to\infty} \frac{1}{R-1} \int_1^R \partial_\nu u_{1,R'}(t,x)\, dR'$$
$$= \mathrm{e}^{ikt} \sum_{l=1}^{l_0} \frac{1}{\lambda - \lambda_l - k^2} \left(\int_{\partial M_0} h_1(y) \partial_\nu \varphi_l(y)\, \mathrm{d}S(y) \right) \partial_\nu \varphi_l(x)$$
$$= \mathrm{e}^{ikt} \Lambda^{Tr}_{g_0,q_0}(\lambda - k^2) h$$

if h_1 is the low frequency component of h.

This proves the averaging formula.

Outline

The geodesic ray transform

The unit sphere bundle :

$$SM_0 = \bigcup_{x \in M_0} S_x, \qquad S_x = \big\{(x,\xi) \in T_x M_0 \,;\, |\xi|_g = 1\big\}.$$

Boundary: $\partial(SM_0) = \{(x,\xi) \in SM_0 \,;\, x \in \partial M_0\}$ union of inward and outward pointing vectors:

$$\partial_\pm(SM_0) = \big\{(x,\xi) \in SM_0 \,;\, \pm\langle \xi, \nu \rangle \leq 0\big\}.$$

Denote by $t \mapsto \gamma(t,x,\xi)$ the unit speed geodesic starting at x in direction ξ, and let $\tau(x,\xi)$ be the time when this geodesic exits M_0.

Geodesic ray transform :

$$T_0 f(x,\xi) = \int_0^{\tau(x,\xi)} f(\gamma(t,x,\xi))\,\mathrm{d}t, \qquad (x,\xi) \in \partial_+(SM_0).$$

Simple manifolds

Definition

A compact manifold (M_0, g_0) with boundary is simple if for any $p \in M_0$ the exponential map $\exp_p$ with its maximal domain of definition is a diffeomorphism onto M_0, and if ∂M_0 is strictly convex (that is, the second fundamental form of $\partial M_0 \hookrightarrow M_0$ is positive definite).

1. Simple manifolds are non-trapping.
2. Simple manifolds are diffeomorphic to a ball.
3. A hemisphere is not simple.

Injectivity of the ray transform

Injectivity of the ray transform is known to hold for

1. Simple manifolds of any dimension.
2. Manifolds of dimension ≥ 3 that have strictly convex boundary and are globally foliated by strictly convex hypersurfaces (Uhlmann-Vasy).
3. A class of non-simple manifolds of any dimension such that there are sufficiently many geodesics without conjugate points and the metric is close to a real-analytic one (Stefanov-Uhlmann).
4. There are counter-examples to injectivity of the ray transform.

Main results in a conformal class

Theorem

Let (M, g) be a compact manifold with boundary which can be embedded in the cylinder T. Let $q_1, q_2 \in C(M)$ such that 0 is not a Dirichlet eigenvalue of the corresponding Schrödinger operators. Assume in addition that the ray transform in the transversal manifold is injective. If $\Lambda_{g,q_1} = \Lambda_{g,q_2}$, then $q_1 = q_2$.

Corollary

Let (M, g) be a compact manifold with boundary which can be embedded in the cylinder T.. Assume in addition that the ray transform in the transversal manifold is injective. If $\Lambda_{cg} = \Lambda_g$, then $c = 1$.

A density property

Green's formula and the fact that $\Lambda_{g,q}^* = \Lambda_{g,\bar{q}}$ yield

$$\int_M (q_1 - q_2) u_1 \overline{u_2}\, \mathrm{d}V = \int_{\partial M} (\Lambda_{g,q_1} - \Lambda_{g,q_2}) u_1 \overline{u_2}\, \mathrm{d}S = 0$$

for all pairs (u_1, u_2) of solutions of the Schrödinger equations $(-\Delta_g + q_1)u_1 = 0$, $(-\Delta_g + \overline{q_2})u_2 = 0$. So we are reduced to the following density property

Is the linear span of products $u_1\overline{u_2}$ of solutions u_1, u_2 to two Schrödinger equations with two potentials q_1 and $\overline{q_2}$ dense in, say, $L^1(M)$?

Thank you for your attention!