Guided waves and Carleman estimates

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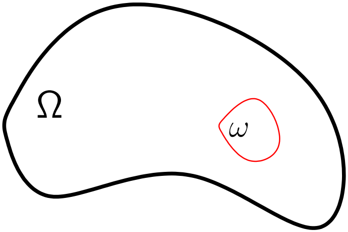
Theorem

There exist 3 strictly positive constants C, λ_0, s_0 *and a function* φ *where* $\beta \in \mathscr{C}^0(\bar{\Omega})$, *such that*

- *without time* $\varphi = e^{\lambda\beta}$

$$\underbrace{s\lambda^2\|e^{s\varphi}\varphi^{\frac{1}{2}}\nabla u\|^2_{L^2(\Omega)} + s^3\lambda^4\|e^{s\varphi}\varphi^{\frac{3}{2}}u\|^2_{L^2(\Omega)}}_{\|u\|^2_{\Omega,\varphi}}$$

$$\leq C\left(\|e^{s\varphi}Au\|^2_{L^2(\Omega)} + s^3\lambda^4\|e^{s\varphi}\varphi^{\frac{3}{2}}u\|^2_{L^2(\omega)}\right) \quad (1)$$



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1. **Result wellknown in lack of interface**
2. There is a correspondance mnemonic between the exponents of s and φ
3. Suppose $u = 0$ on ω and $Au = 0$. Then this theorem implies that $u = 0$ on Ω. That is an uniqueness result.
4. Suppose $Au = \mu u$ (u being an eigenfunction) then

$$\begin{aligned} s^3\lambda^4\|e^{s\varphi}\varphi^{\frac{3}{2}}u\|^2_{L^2(\Omega)} &\leq C(\mu^2\|e^{s\varphi}u\|^2_{L^2(\Omega)} + s^3\lambda^4\|e^{s\varphi}\varphi^{\frac{3}{2}}u\|^2_{L^2(\omega)}) \\ &\leq C(\mu^2\|e^{s\varphi}\varphi^{\frac{3}{2}}u\|^2_{L^2(\Omega)} + s^3\lambda^4\|e^{s\varphi}\varphi^{\frac{3}{2}}u\|^2_{L^2(\omega)}) \end{aligned}$$

$$\Longrightarrow \left(1 - \frac{C\mu^2}{s^3\lambda^4}\right)\|e^{s\varphi}\varphi^{\frac{3}{2}}u\|^2_{L^2(\Omega)} \leq C\|e^{s\varphi}\varphi^{\frac{3}{2}}u\|^2_{L^2(\omega)}$$

If we choose s and λ large enough, we see that we control u on Ω by its restriction on ω.

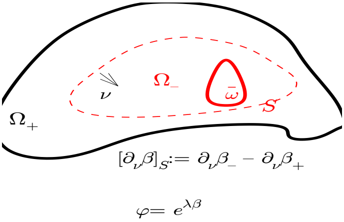


Figure: an example considered by Doubova-Osses & Puel, 2002

Theorem

There exist 3 strictly positive constants C, λ_0, s_0 and a function φ where $\beta \in \mathscr{C}^0(\bar{\Omega})$, such that

- *without time* $\varphi = e^{\lambda\beta}$ *($A = -\nabla \cdot c\nabla$)*

$$\underbrace{s\lambda^2\|e^{s\varphi}\varphi^{\frac{1}{2}}\nabla u\|^2_{L^2(\Omega)} + s^3\lambda^4\|e^{s\varphi}\varphi^{\frac{3}{2}}u\|^2_{L^2(\Omega)}}_{\|u\|^2_{\Omega,\varphi}}$$
$$+ s\lambda\left(|e^{s\varphi}\varphi^{\frac{1}{2}}\nabla_\tau u|^2_{L^2(S)} + |e^{s\varphi}\varphi^{\frac{1}{2}}\partial_{x_n}u_{|S^\pm}|^2_{L^2(S)}\right) + s^3\lambda^3|e^{s\varphi}\varphi^{\frac{3}{2}}u_{|S}|^2_{L^2(S)}$$
$$\leq C\left(\|e^{s\varphi}Au\|^2_{L^2(\Omega)} + s^3\lambda^4\|e^{s\varphi}\varphi^{\frac{3}{2}}u\|^2_{L^2(\omega)}\right) \quad (2)$$

$\forall u \in D(A), s \geq s_0, \lambda \geq \lambda_0.$

- with time $\varphi = \frac{e^{\lambda\beta}}{t(T-t)}, 0 < t < T, (A = \partial_t + \nabla \cdot c\nabla)$

$$s\lambda^2\left\|\frac{e^{-s\eta}}{t^{1/2}(T-t)^{1/2}}\varphi^{\frac{1}{2}}|\nabla q|\right\|^2 + s^3\lambda^4\left\|\frac{e^{-s\eta}}{t^{3/2}(T-t)^{3/2}}\varphi^{\frac{3}{2}}|q|\right\|^2$$
$$\leq C\left(\left\|e^{-s\eta}|\partial_t + \mathrm{div}\,(c\nabla q)|\right\|^2 + s^3\lambda^4\int_{\omega\times(0,T)}\frac{e^{-2s\eta}}{t^3(T-t)^3}\varphi^3|q|^2\right) \quad (3)$$

With S some difficulties appear:

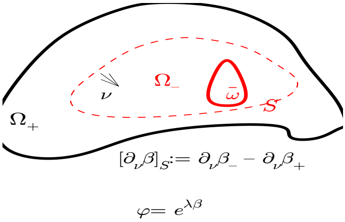
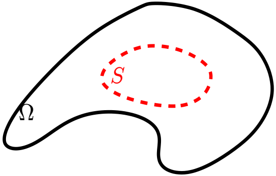


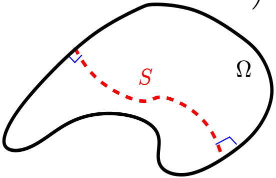
Figure: an example considered by Doubova-Osses & Puel, 2002

What the sign of

$$\int_S \left(|s\lambda\varphi u|^2 [c^2(\partial_\nu \beta)^3]_S + [|\partial_\nu u|^2 c^2 \partial_\nu \beta]_S - |\nabla_\tau u|^2 \| [\partial_\nu \beta c^2]_S \| \right) e^{s\varphi} d\sigma \text{?}$$



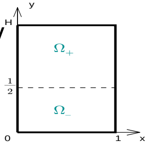
Le Rousseau-Robbiano



Benabdallah-Dermenjian-Thevenet

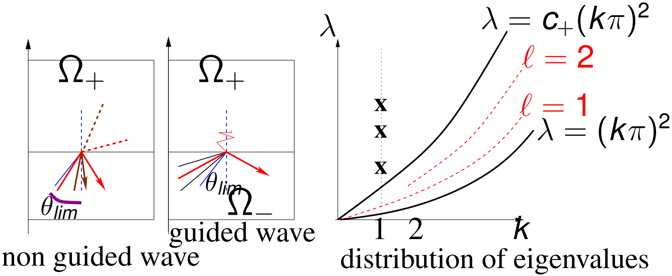
Guided Waves: a disturbing situation

$A = -\nabla\cdot(c\nabla)$, $\Omega = (0,1)\times(0,H)$, DC on boundary



$$c(x,y) = c(y) = \begin{cases} c_+ & h < y \le H, \\ 1 & 0 \le y < h. \end{cases}$$

$1 < c_+$



- non guided wave: $\lambda_{k,\ell} > c_+(k\pi)^2$

$$u_{k,\ell}(x,y) = \sin(k\pi x)\begin{cases} \sin(py) \text{ if } 0 < y < h, \\ \frac{\sin(ph)}{\sin(q(h-H))}\sin(q(z-H)) \text{ if } h < y < H. \end{cases}$$

where $p^2 + (k\pi)^2 = c_+(q^2 + (k\pi)^2) = \lambda_{k,\ell}, \quad \dfrac{\tan(q(H-h))}{q} = -\dfrac{\tan(ph)}{p}.$

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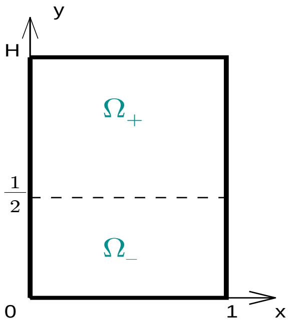
These eigenfunctions decrease exponentially to 0 far from the interface, in the part where the diffusion coefficient c is the biggest.

$$c(x,y) = c(y) = \begin{cases} c_+ & h < y \leq H, \\ 1 & 0 \leq y < h. \end{cases}$$

$\Omega = (0,1) \times (0,H)$

$\Omega_+ = (0,1) \times (h,H)$



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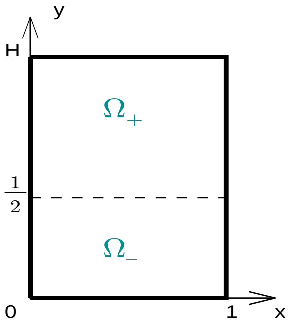

For $c_+ > 1$ and fixed ℓ, a guided wave satisfied

$$\frac{\int_{\Omega_+} |u_{k,\ell}(x,y)|^2 \, dx\, dy}{\int_{\Omega} |u_{k,\ell}(x,y)|^2 \, dx\, dy} = \frac{(H-h)p^2}{2\tilde{q}^2} \cos^2(ph) \left[\tanh^2(\tilde{q}(H-h)) + \frac{\tanh(\tilde{q}(H-h))}{\tilde{q}(H-h)} - 1 \right]$$

Energy is concentrated in the "valley". Are there lost informations ?

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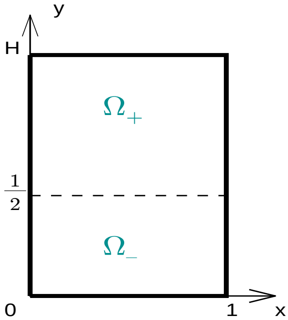
$c_+ > 1, \varphi = e^{\lambda\beta}.$



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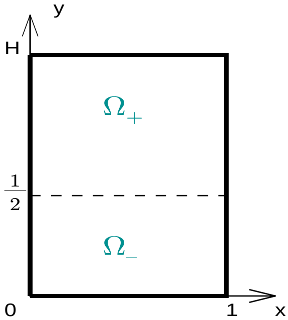


$$(*) \int_S \left(|s\lambda\varphi u|^2 [c^2(\partial_y\beta)^3]_S + [|\partial_y u|^2 c^2 \partial_y \beta]_S - |\partial_x u|^2 \|[\partial_y \beta c^2]_S\| \right) e^{s\varphi} d\sigma$$

ratio of amplitudes $\frac{\partial_x u_{k,\ell}}{\partial_y u_{k,\ell}}(x,h) = \frac{k\pi}{p} \frac{\tan(ph)}{\tan(k\pi x)}, 0 < x < 1.$

ratio of norms $\frac{\|\partial_x u_{k,\ell}\|_{L^2(S)}}{\|\partial_y u_{k,\ell}\|_{L^2(S)}} = k\pi \left| \frac{\tan(ph)}{ph} \right|$

$$\int_S \left(|s\lambda\varphi u|^2 [c^2(\partial_y\beta)^3]_S + [|\partial_y u|^2 c^2 \partial_y \beta]_S - |\partial_x u|^2 \|[\partial_y \beta c^2]_S\| \right) e^{s\varphi} d\sigma$$

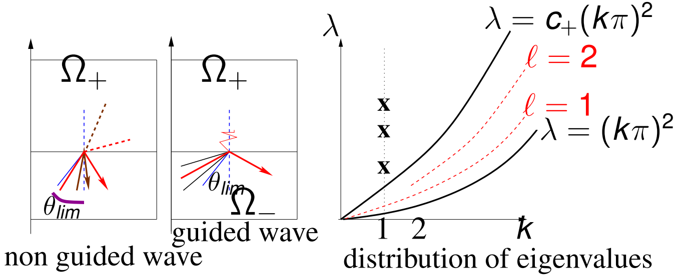
Ratio of norms

- Guided wave

$$\sqrt{\frac{c_+}{c_+-1}} \tanh\left(k\pi\sqrt{\frac{c_+-1}{c_+}}(H-h)\right) \leq \frac{\|\partial_x u_{k,\ell}\|_{L^2(S)}}{\|\partial_y u_{k,\ell}\|_{L^2(S)}} \leq k\pi(H-h).$$

- Non guided wave

$\frac{\|\partial_x u_{k,\ell}\|_{L^2(S)}}{\|\partial_y u_{k,\ell}\|_{L^2(S)}}$ bounded



$$\Downarrow$$

Necessity of an alternative approach for the guided waves

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$$-\Delta u = \frac{1}{c} f.$$

$$\begin{cases} \partial_x^2 u + \partial_y^2 u = -\frac{f}{c}, \quad y \neq 0, \\ [u]_S = 0, [c\partial_y u]_S = 0 \end{cases}$$

$$\Downarrow$$

Necessity of an alternative approach for the guided waves

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"Partial Fourier Transform" en x gives this ODE

$$-\partial_y \hat{u}_k + \mu_k^2 \hat{u}_k = \frac{1}{c} f_k \tag{4}$$

Lemma (Calderòn)

The solution of

$$\begin{cases} \nu^{''} - \mu^2 \nu = F \quad s \in (-\delta, 0) \cup (0, \delta), \\ \nu(-\delta) = \nu(\delta) = 0, \quad \nu(0_-) = \nu(0_+), \quad c_+ \nu'(0_+) = c_- \nu'(0_-) + \theta \end{cases}$$

satisfies

$$\nu(0) = -\frac{1}{\mu} \int_0^\delta \frac{\sinh(\mu(\delta - y))}{(c_+ + c_-)\cosh(\mu\delta)} (c_+ F(y) + c_- F(-y)) dy - \frac{\theta \tanh(\mu\delta)}{\mu(c_+ + c_-)}.$$

$$W_k(y) := \frac{1}{2} s\lambda\varphi_{|S} e^{2s\varphi_{|S}} |\hat{u}_k(y)|^2$$

$$\sum_k \mu_k^2 W_k \sim \|\partial_x u\|^2$$

$$\begin{cases} \partial_y^2 W_k - \mu_k^2 W_k = -d_k \\ W_k(\pm H) = 0, W_k(0^+) = W_k(0^-) \text{ and } c_+^j W_k'(0^+) = c_-^j W_k'(0^-) \\ -d_k = -\frac{f_k}{c} \hat{u}_k s\lambda\varphi_{|S} e^{2s\varphi_{|S}} + \gamma\mu_k^2 W_k \\ \qquad\qquad\qquad +(1-\gamma)\mu_k^2 W_k + (\partial_y \hat{u}_k)^2 s\lambda\varphi_{|S} e^{2s\varphi_{|S}} \end{cases}$$

$$W_k(0) = \frac{1}{\mu_k} \int_0^H \frac{\sinh(\mu_k(H-y))}{(c_+ + c_-)\cosh(\mu_k H)} \left(c_+ d_k(y) + c_- d_k(-y)\right) \mathrm{d}y$$

It’s the end.