Lipschitz stability of the conductivity coefficient as a function of the resolvent

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1. M. Choulli and E. Zuazua, Lipschitz stability of the conductivity coefficient as a function of the resolvent for scalar elliptic problems, preprint.

Summary

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Formulation of many design problems in engineering :

optimal control problems with parametric PDE constraints

$\rightarrow$ frequent numerical solution of a PDE depending on dynamically updated parameters

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Model situation : $(H, \|\cdot\|)$ Hilbert space, $D \subset \mathbb{R}^d$ compact and $(a_\mu)_{\mu \in D}$ a family of symmetric, continuous, and elliptic bilinear forms

$$a_\mu : H \times H \rightarrow \mathbb{R},$$

so that $\|\cdot\|_\mu = a_\mu(\cdot,\cdot)^{1/2}$ are uniformly equivalent to $\|\cdot\|$:

$$c\|u\| \leq \|u\|_\mu \leq C\|u\|, \quad u \in H, \ \mu \in D.$$

Typical optimal control problem

$$\min_{\mu \in D} J(u_\mu),$$

for some functional J, where $u_\mu \in H$ is the solution of the variational problem

$$a_\mu(u_\mu, v) = \Phi(v), \quad v \in H, \tag{1}$$

with $\Phi \in H'$.

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→ Employing a standard highly accurate numerical solution of the PDE for each parameter value $\mu \in D$ is "infeasible".

→ Remedy : exploit the compactness of the set

$$\mathcal{F} := \{u_\mu;\ \mu \in D\}$$

in the energy space H.

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$(f_0, \ldots, f_{m-1})$ is called reduced basis.

$\rightarrow$ Key question : find "good" parameters μ_j or, equivalently, good basis functions $f_j \in \mathcal{F}$: greedy stategy.

$\rightarrow$ Problem : find functions $\{f_0, \ldots, f_{m-1}\}$ so that each $f \in \mathcal{F}$ is well approximated by the elements of the subspace $F_n := \text{span}\{f_0, \ldots, f_{m-1}\}$.

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Greedy algorithm :

$$f_0 = \text{argmax}_{f \in \mathcal{F}} \|f\|, \quad F_1 = \text{span}\{f_0\}.$$

If $f_0, \dots, f_{m-1}$ have been chosen, $F_m = \text{span}\{f_0, \dots, f_{m-1}\}$ and P_m is the projector onto F_m, we choose f_m as

$$f_m = \text{argmax}_{f \in \mathcal{F}} \|f - P_m f\|.$$

"Computationally not feasible" $\rightarrow$ weak greedy algorithm

Strategy : substitute the error $\|f - P_m f\|$ by a "surrogate" $r_m(f)$ satisfying

$$c_* r_m(f) \leq \|f - P_m f\| \leq c^* r_m(f), \quad f \in H.$$

In other words, we seek f_m so that

$$f_m = \text{argmax}_{f \in \mathcal{F}} r_m(f).$$

It is worth mentioning that

$$\|f_m - P_m f_m\| \geq \gamma \max_{f \in \mathcal{F}} \|f - P_m f\|, \quad \gamma = \frac{c_*}{c^*} \in (0, 1).$$

Let $R_\mu \in \mathscr{B}(H', H)$ be the resolvent associated to the variational problem (1)

$$R_\mu : \Phi \in H' \to R_\mu \Phi := u_\mu \in H.$$

Objective : provide greedy and weak greedy algorithms independent of the given source terms.

Roughly, we need to carry out greedy algorithms when the preceding $\mathcal{F}$ is substituted by the following one

$$\mathcal{F} = \{R_\mu ;\ \mu \in D\}.$$

Two references :

P. Binev, A. Cohen, W. Dahmen, R. De Vore, G. Petrova and P. Wojtaszczyk,
Convergence rates for greedy algorithms in reduced basis methods,
SIAM J. Math. Anal. 43 (3) (2011), 1457-1472.

A. Buffa, Y. Maday, A. T. Patera, C. Prud'homme and G. Turinici,
A Priori convergence of the greedy algorithm for the parameterized reduced basis,
Math. Model. Numer. Anal. 46 (2012), 595-603.

Summary

Let $\Omega \subset \mathbb{R}^n$ bounded [2], $n \geq 1$. Fix $0 < \sigma_0 < \sigma_1$ and set

$$\Sigma = \{\sigma \in L^\infty(\Omega);\ \sigma_0 \leq \sigma \leq \sigma_1 \text{ a.e. in } \Omega\}.$$

As usual $H_0^1(\Omega)$ is endowed with the norm

$$\|w\|_{H_0^1(\Omega)} = \|\nabla w\|_{L^2(\Omega)^n}.$$

2. It is enough to assume that Ω has the Poincaré inequality.

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Denote by $\langle \cdot, \cdot \rangle_{-1,1}$ the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Let $f \in H^{-1}(\Omega)$ and $\sigma \in \Sigma$. By Lax-Milgram's lemma the variational problem

$$\int_\Omega \sigma \nabla u \cdot \nabla v = \langle f, v \rangle_{-1,1}, \quad v \in H_0^1(\Omega),$$

has a unique solution $u_\sigma \in H_0^1(\Omega)$. Moreover,

$$\|u_\sigma\|_{H_0^1(\Omega)} \leq \sigma_0^{-1} \|f\|_{H^{-1}(\Omega)}.$$

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2. It is enough to assume that Ω has the Poincaré inequality.

Whence the bounded operator, where $\sigma \in \Sigma$,

$$A_\sigma : H_0^1(\Omega) \to H^{-1}(\Omega) : A_\sigma u = -\mathrm{div}(\sigma \nabla u)$$

has an inverse $R_\sigma := A_\sigma^{-1} \in \mathscr{B}(H^{-1}(\Omega), H_0^1(\Omega))$.

The norm of $\mathscr{B}(H^{-1}(\Omega), H_0^1(\Omega))$ is denoted by $\| \cdot \|_{-1,1}$.

Theorem 1

For any $\sigma, \widetilde{\sigma} \in \Sigma$,

$$\sigma_0^2 \| R_\sigma - R_{\widetilde{\sigma}} \|_{-1,1} \le \| \sigma - \widetilde{\sigma} \|_{L^\infty(\Omega)} \le \sigma_1^2 \| R_\sigma - R_{\widetilde{\sigma}} \|_{-1,1}. \tag{2}$$

If d_∞ be the distance induced by the L^∞ norm, inequality (2) in Theorem 1 can rephrased as

$$\sigma_0^2 d_R \leq d_\infty \leq \sigma_1^2 d_R \ \text{ on } \Sigma \times \Sigma,$$

where d_R is the metric on Σ defined as follows

$$d_R(\sigma, \widetilde{\sigma}) = \|R_\sigma - R_{\widetilde{\sigma}}\|_{-1,1}, \ \ \sigma, \widetilde{\sigma} \in \Sigma.$$

The first inequality in (2) is contained in the following lemma.

Lemma 2

For any $\sigma, \widetilde{\sigma} \in L^\infty(\Omega)$ *satisfying* $\sigma_0 \leq \sigma, \widetilde{\sigma}$,

$$\|R_\sigma - R_{\widetilde{\sigma}}\|_{-1,1} \leq \sigma_0^{-2} \|\sigma - \widetilde{\sigma}\|_{L^\infty(\Omega)}.$$

This lemma is proved in a straightforward manner by using energy estimates.

The proof of the second inequality in (2) is based on

Lemma 3

Let $\gamma \in L^\infty(\Omega)$. For a.e. $x_0 \in \Omega$, there exists a sequence $(u_{x_0,\epsilon})$ in $H_0^1(\Omega)$ so that $\|u_{x_0,\epsilon}\|_{H_0^1(\Omega)} = 1$, for each ϵ, and

$$\lim_{\epsilon} \int_\Omega \gamma(x)|\nabla u_{x_0,\epsilon}|^2 dx = \gamma(x_0).$$

Corollary 4

Let $\gamma \in L^\infty(\Omega)$ *so that*

$$\int_\Omega \pm\gamma|\nabla u|^2 dx \leq C, \quad \text{for any } u \in H_0^1(\Omega),\ \|u\|_{H_0^1(\Omega)} = 1,$$

for some constant $C > 0$. *Then*

$$\|\gamma\|_{L^\infty(\Omega)} \leq C. \tag{3}$$

Let $\sigma, \widetilde{\sigma} \in \Sigma_0$. From the identity

$$A_\sigma - A_{\widetilde{\sigma}} = A_\sigma (R_{\widetilde{\sigma}} - R_\sigma) A_{\widetilde{\sigma}},$$

we get

$$\|A_\sigma - A_{\widetilde{\sigma}}\| \leq \sigma_1^2 \|R_\sigma - R_{\widetilde{\sigma}}\|. \tag{4}$$

On the other hand

$$\langle (A_\sigma - A_{\widetilde{\sigma}}) u, v \rangle_{-1,1} = \int_\Omega (\sigma - \widetilde{\sigma}) \nabla u \cdot \nabla v dx, \quad u, v \in H_0^1(\Omega),$$

implying

$$\int_\Omega (\sigma - \widetilde{\sigma}) \nabla u \cdot \nabla v dx \leq \|A_\sigma - A_{\widetilde{\sigma}}\| \|\|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}, \quad u, v \in H_0^1(\Omega).$$

The last inequality and (4) entail

$$\int_\Omega \pm(\sigma - \widetilde{\sigma})|\nabla u|^2 dx \leq \sigma_1^2 \|R_\sigma - R_{\widetilde{\sigma}}\|, \quad u \in H_0^1(\Omega),\ \|u\|_{H_0^1(\Omega)} = 1$$

which yields the second inequality of (2) by Corollary 4.

Summary

The Neumann case

For $\sigma \in \Sigma$, define $A_\sigma^N : H^1(\Omega) \to (H^1(\Omega))'$ by

$$\langle A_\sigma^N u, v\rangle := \int_\Omega \sigma \nabla u \cdot \nabla v dx + \int_\Omega uv dx, \quad u, v \in H^1(\Omega),$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$.
A_σ^N is bounded and, with $\underline{\sigma}_1 = \max(\sigma_1, 1)$,

$$\|A_\sigma^N u\|_{(H^1(\Omega))'} \leq \underline{\sigma}_1 \|u\|_{H^1(\Omega)}, \quad u \in H^1(\Omega).$$

If $f \in (H^1(\Omega))'$, we get by applying Lax-Milgram's lemma that the variational problem

$$\int_\Omega \sigma \nabla u_\sigma \cdot \nabla v dx + \int_\Omega u_\sigma v dx = \langle f, v \rangle, \quad v \in H^1(\Omega) \tag{5}$$

has a unique solution $u_\sigma \in H^1(\Omega)$.

Whence $A_\sigma^N u_\sigma = f$ and $v = u_\sigma$ in (5) implies

$$\|u_\sigma\|_{H^1(\Omega)} \leq \underline{\sigma}_0^{-1} \|f\|_{(H^1(\Omega))'}, \quad \text{with } \underline{\sigma}_0 = \min(\sigma_0, 1). \tag{6}$$

Thus A_σ^N has a bounded inverse

$$R_\sigma^N := (A_\sigma^N)^{-1} : (H^1(\Omega))' \to H^1(\Omega).$$

As

$$\langle (A_\sigma^N - A_{\widetilde{\sigma}}^N)u, v\rangle = \int_\Omega (\sigma - \widetilde{\sigma})\nabla u \cdot \nabla v dx, \quad u, v \in H^1(\Omega),$$

we get similarly to Theorem 1

$$\|\sigma - \widetilde{\sigma}\|_{L^\infty(\Omega)} \leq \underline{\sigma}_1^2 \|R_\sigma^N - R_{\widetilde{\sigma}}^N\|_{-1,1}.$$

On the other hand,

$$\underline{\sigma}_0^2 \|R_\sigma^N - R_{\widetilde{\sigma}}^N\|_{-1,1} \leq \|\sigma - \widetilde{\sigma}\|_{L^\infty(\Omega)}.$$

In other words, we have

$$\underline{\sigma}_0^2 \|R_\sigma^N - R_{\widetilde{\sigma}}^N\|_{-1,1} \leq \|\sigma - \widetilde{\sigma}\|_{L^\infty(\Omega)} \leq \underline{\sigma}_1^2 \|\sigma - \widetilde{\sigma}\|_{L^\infty(\Omega)}.$$

The Robin case

Assume that Ω has Lipschitz boundary Γ.

Pick $\beta \in L^\infty(\Gamma)$ so that $\beta \geq 0$ and $\beta \geq \beta_0$ on an open subset Γ_0 of Γ, where $\beta_0 > 0$ is some constant. Consider the Robin BVP

$$-\operatorname{div}(\sigma \nabla u) = f \text{ in } \Omega \;\text{ and }\; \sigma \partial_\nu u + \beta u = 0 \text{ on } \Gamma, \tag{7}$$

where $\partial_\nu = \nu \cdot \nabla$ with ν the exterior normal unit normal vector field on Γ.

If $\sigma \in \Sigma$, define $A_\sigma^R : H^1(\Omega) \to (H^1(\Omega))'$ by

$$\langle A_\sigma^R u, v\rangle := \int_\Omega \sigma \nabla u \cdot \nabla v dx + \int_\Gamma \beta uv dS(x), \quad u, v \in H^1(\Omega).$$

Equip $H^1(\Omega)$ with the norm

$$\|u\|_{H^1(\Omega)} = \left(\|\nabla u\|^2_{L^2(\Omega)^n} + \|u\|^2_{L^2(\Gamma_0)}\right)^{1/2}. \tag{8}$$

A_σ^R is bounded and

$$\|A_\sigma^R u\|_{(H^1(\Omega))'} \leq \underline{\sigma}_1 \|u\|_{H^1(\Omega)}, \quad u \in H^1(\Omega), \quad \text{with } \underline{\sigma}_1 = \max(\sigma_1, \kappa\|\beta\|_{L^\infty(\Gamma)}),$$

where κ is the norm of the trace operator $u \in H^1(\Omega) \to u_{|\Gamma} \in L^2(\Gamma)$ when $H^1(\Omega)$ is endowed with the norm (8).

Consider the bilinear form

$$a(u,v)=\int_\Omega \sigma\nabla u\cdot\nabla v dx+\int_\Gamma \beta uvdS(x),\quad u,v\in H^1(\Omega).$$

$u\to a(u,u)$ defines a norm on $H^1(\Omega)$ equivalent to the usual norm on $H^1(\Omega)$. Let $f\in(H^1(\Omega))'$. By Riesz's representation theorem, there exists a unique $u_\sigma\in H^1(\Omega)$ satisfying

$$a(u_\sigma,\psi)=\int_\Omega \sigma\nabla u_\sigma\cdot\nabla\psi dx+\int_\Gamma \beta u_\sigma\psi dS(x)=\langle f,\psi\rangle,\quad \psi\in H^1(\Omega). \quad (9)$$

Note that u_σ is nothing but the variational solution of the BVP (7).

From (9),

$$\|u_\sigma\|_{H^1(\Omega)} \leq \underline{\sigma}_0 \|f\|_{(H^1(\Omega))'}, \quad \text{with } \underline{\sigma}_0 = \min(\sigma_0, \beta_0).$$

Consequently, A_σ^R possesses a bounded inverse

$$R_\sigma^R = (A_\sigma^R)^{-1} : (H^1(\Omega))' \to H^1(\Omega)$$

defined by $R_\sigma^R f := u_\sigma$ for $f \in (H^1(\Omega))'$.

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Starting from

$$\langle (A_\sigma^R - A_{\widetilde{\sigma}}^R)u, v\rangle = \int_\Omega (\sigma - \widetilde{\sigma})\nabla u \cdot \nabla v dx, \quad u, v \in H^1(\Omega),$$

we get similarly to the Neumann case

$$\underline{\sigma}_0^2 \|R_\sigma^R - R_{\widetilde{\sigma}}^R\|_{-1,1} \leq \|\sigma - \widetilde{\sigma}\|_{C(\overline{\Omega})} \leq \underline{\sigma}_1^2 \|R_\sigma^R - R_{\widetilde{\sigma}}^R\|_{-1,1},$$

Summary

Ω is a C^2-smooth bounded domain of $\mathbb{R}^n$, $n \geq 2$, diffeomorphic to the unit ball of $\mathbb{R}^n$, and $\Gamma = \partial\Omega$.

Let $\sigma \in \Sigma$. For $g \in H^{\frac{1}{2}}(\Gamma)$ [3], denote by $u_\sigma \in H^1(\Omega)$ the unique weak solution of the BVP

$$\mathrm{div}(\sigma \nabla u) = 0 \text{ in } \Omega \quad \text{and} \quad u = g \text{ on } \Gamma.$$

We prove

$$\|u_\sigma\|_{H^1(\Omega)} \leq (1 + \sigma_0^{-1}\sigma_1)\|g\|_{H^{\frac{1}{2}}(\Gamma)}.$$

3. Here $H^{\frac{1}{2}}(\Gamma)$ is viewed as the quotient space $H^1(\Omega)/H_0^1(\Omega)$.

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We prove

$$\|u_\sigma\|_{H^1(\Omega)} \leq (1 + \sigma_0^{-1}\sigma_1)\|g\|_{H^{\frac{1}{2}}(\Gamma)}.$$

Then $\mathcal{R}_\sigma$ given by $\mathcal{R}_\sigma g := u_\sigma$ defines a bounded operator from $H^{\frac{1}{2}}(\Gamma)$ into $H^1(\Omega)$ and

$$\|\mathcal{R}_\sigma\|_{\frac{1}{2},1} \leq 1 + \sigma_0^{-1}\sigma_1.$$

Here $\|\cdot\|_{\frac{1}{2},1}$ denotes the norm in $\mathscr{B}(H^{\frac{1}{2}}(\Gamma), H^1(\Omega))$.

3. Here $H^{\frac{1}{2}}(\Gamma)$ is viewed as the quotient space $H^1(\Omega)/H^1_0(\Omega)$.

Fix $\overline{g} \in C^2(\Gamma)$ so that

$$\Gamma_- = \{x \in \Gamma;\ \overline{g}(x) = \min \overline{g}\} \quad \Gamma_+ = \{x \in \Gamma;\ \overline{g}(x) = \max \overline{g}\}$$

are nonempty and connected, and the following condition fulfills : there exists a continuous non decreasing function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ and $\rho_0 > 0$ so that, for any $0 < \rho \leq \rho_0$,

$$|\nabla_\tau \overline{g}| \geq \psi(\rho), \text{ on } \{x \in \Gamma;\ \mathrm{dist}(x, \Gamma_- \cup \Gamma_+) \geq \rho\}.$$

where ∇_τ denotes the tangential gradient.

Such a function is called quantitatively unimodal.

For $\sigma_1 > \sigma_0$, define

$$\mathcal{E} = \{\sigma \in W^{1,\infty}(\Omega);\ \sigma_0 \leq \sigma \text{ and } \|\sigma\|_{W^{1,\infty}(\Omega)} \leq \sigma_1\}.$$

Theorem 5

[a]There exist two constants $C > 0$ and $\gamma > 0$, that can depend on Ω, $\mathcal{E}$ and $\overline{g}$, so that

$$\|\sigma - \widetilde{\sigma}\|_{L^\infty(\Omega)} \leq C\|\mathcal{R}_\sigma \overline{g} - \mathcal{R}_{\widetilde{\sigma}} \overline{g}\|^{\gamma}_{L^2(\Omega)},\quad \sigma, \widetilde{\sigma} \in \mathcal{E}_0,$$

where $\mathcal{E}_0 = \{\sigma \in \mathcal{E};\ \sigma = \overline{\sigma} \text{ on } \Gamma\}$, for some fixed $\overline{\sigma} \in \mathcal{E}$.

a. G. Alessandrini, M. Di Cristo, E. Francini and S. Vessella, Stability for quantitative photoacoustic tomography with well chosen illuminations, arXiv :1505.03657.

Corollary 6

There exist two constants $C > 0$ and $\gamma > 0$, that can depend on Ω and $\mathcal{E}$, so that

$$\|\sigma - \widetilde{\sigma}\|_{L^\infty(\Omega)} \leq C \|\mathcal{R}_\sigma - \mathcal{R}_{\widetilde{\sigma}}\|^{\gamma}_{\frac{1}{2},1}, \quad \sigma, \widetilde{\sigma} \in \mathcal{E}_0,$$

where $\mathcal{E}_0$ is as in the preceding theorem.

This result can be interpreted as a Hölder stability estimate of determining σ from $\mathcal{R}_\sigma$.

Summary

Consider the BVP

$$-(\sigma(x)u_x)_x = f \text{ in } (0,1) \quad u_x(0) = 0 \quad \text{and} \quad u(1) = 0. \tag{10}$$

Recall

$$\Sigma = \{\sigma \in L^\infty((0,1));\ \sigma_0 \leq \sigma \leq \sigma_1 \text{ a.e. in } (0,1)\}$$

and let $H = \{u \in H^1((0,1));\ u(1) = 0\}$ equipped with the norm

$$\|u\|_H = \|u_x\|_{L^2((0,1))}.$$

By Lax-Milgram's lemma or Riesz's representation theorem, for each $f \in H'$, there exists a unique $u = u_\sigma \in H$ so that

$$\int_0^1 \sigma(x)u_x(x)v_x(x)dx = \langle f, v\rangle, \quad v \in H,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between H and its dual H'. Note that u_σ is nothing but the variational solution of the BVP (10).

Therefore $R_\sigma : f \in H' \to u_\sigma \in H$ defines a bounded operator with

$$\|R_\sigma f\|_H \leq \sigma_0^{-1} \|f\|_{H'}.$$

Pick $f \in L^2((0,1))$ and set

$$v(x) = \int_x^1 \frac{1}{a(t)} \int_0^t f(s)dsdt, \quad x \in [0,1].$$

Then v is absolutely continuous, $v(1) = 0$ and

$$v_x(x) = -\frac{1}{\sigma(x)} \int_0^x f(t)dt \ \text{ a.e. } (0,1). \tag{11}$$

On the other hand, if $w \in H$, we get by applying Green's formula

$$\int_0^1 \sigma(x) v_x(x) w_x(x) = -\int_0^1 w_x(x) \left(\int_0^x f(t)dt \right) dx = \int_0^1 w(x) f(x) dx.$$

In other words, $v = R_\sigma f$.

Denote by $W^{-1,1}((0,1))$ the closure of $C_0^\infty((0,1))$ for the norm

$$\|f\|_{W^{-1,1}(0,1)} = \left\| \int_0^x f(t)dt \right\|_{L^1((0,1))}.$$

The norm of $\mathscr{B}(W^{-1,1}((0,1)), L^1((0,1)))$ is denoted by $\|\cdot\|_{-1,1}$

Lemma 7

Let $m \in L^\infty((0,1))$ *and* $T_m : W^{-1,1}((0,1)) \to L^1((0,1))$ *given as follows*

$$T_m f(x) = m(x) \int_0^x f(t)dt, \quad a.e. \;\; x \in (0,1). \tag{12}$$

Then $\|T_m\|_{-1,1} = \|m\|_{L^\infty((0,1))}$.

Introduce the distance between the resolvents

$$\|R_\sigma - R_{\widetilde{\sigma}}\|_* = \left\| T_{1/\sigma} - T_{1/\widetilde{\sigma}} \right\|_{-1,1} \tag{13}$$

For $\sigma, \widetilde{\sigma} \in \Sigma$, formula (11) yields

$$(R_\sigma f - R_{\widetilde{\sigma}} f)_x = \left(\frac{1}{\widetilde{\sigma}} - \frac{1}{\sigma} \right) \int_0^x f(t)dt \ \text{ a.e. in } (0,1).$$

Therefore, we obtain as a consequence of Lemma 7

$$||R_\sigma - R_{\widetilde{\sigma}}||_* = \left\| \frac{1}{\widetilde{\sigma}} - \frac{1}{\sigma} \right\|_{L^\infty((0,1))}.$$

This identity implies

$$\sigma_1^{-2} ||\widetilde{\sigma} - \sigma||_{L^\infty((0,1))} \leq ||R_\sigma - R_{\widetilde{\sigma}}||_* \leq \sigma_0^{-2} ||\widetilde{\sigma} - \sigma||_{L^\infty((0,1))}.$$

Distance to a subspace

Consider a distinguished coefficient τ and $m \geq 2$ others, $\sigma_1, \cdots, \sigma_m$, and denote the corresponding resolvents by R_τ and $R_1, \cdots, R_m$, respectively.

From identity (11), we have

$$\left(R_\tau f - \sum_{i=1}^{m} a_i R_i f \right)_x = \left(\sum_{i=1}^{m} \frac{a_i}{\sigma_i} - \frac{1}{\tau} \right) \int_0^x f(t)dt \text{ a.e. in } (0,1) \quad (14)$$

that yields the representation of the difference of a resolvent with respect to the linear combination of a finite number of others.

Then

$$\left\| R_\tau - \sum_{i=1}^{m} a_i R_i \right\|_* = \left\| \sum_{i=1}^{m} \frac{a_i}{\sigma_i} - \frac{1}{\tau} \right\|_{L^\infty((0,1))} . \quad (15)$$

In other words, the L^∞-distance between inverses of coefficients, yields an adequate "surrogate" for the distance between the resolvents :

$$\mathrm{dist}_*(R_\tau, \mathrm{span}\{R_i, 1 \leq i \leq m\}) = \mathrm{dist}_{L^\infty((0,1))}\left(\tau^{-1}, \mathrm{span}\{\sigma_i^{-1}, 1 \leq i \leq m\}\right).$$

dist_* is the distance associated to $\|\cdot\|_*$-norm in (13).

Greedy algorithm

Consider the parameter-dependent BVP

$$-(\sigma(x,\mu)u_x)_x = f \text{ in } (0,1) \;\; u_x(0)=0 \;\text{ and }\; u(1)=0,$$

with $\mu \in D$, where D is some compact subset of $\mathbb{R}^d$.

Assume that $\sigma(\cdot,\mu) \in \Sigma$, for any $\mu \in D$.

Fix some $0<\gamma<1$. Having found $\mu_1,\ldots,\mu_{m-1}$, with the corresponding diffusivity coefficients $\sigma_1,\ldots,\sigma_{m-1}$, $\sigma_i=\sigma(\cdot,\mu_i)$, we choose the next element μ_m such that the corresponding diffusivity coefficient σ_m satisfies

$$\mathrm{dist}_{L^\infty((0,1))}\left(\sigma_m^{-1}, \mathrm{span}\{\sigma_i^{-1}; i = 1, \ldots, m-1\}\right) \\ \geq \gamma \max_{\mu \in D} \mathrm{dist}_{L^\infty((0,1))}\left(\sigma(\cdot,\mu)^{-1}, \mathrm{span}\{\sigma_i^{-1}; i = 1, \ldots, m-1\}\right). \tag{16}$$

The important consequence of this fact is that, for the identification of the most relevant parameter values μ_n, we do not need to solve the elliptic equation, but simply deal with the family of coefficients $\sigma(x,\mu)$, solving a classical L^∞-minimisation problem in an approximated manner as indicated in (16) by a multiplicative factor ($0 < \gamma < 1$).

Summary

Assume that Ω is $C^{1,1}$.

For $f \in L^2(\Omega)$ and $\rho \in L^\infty(\Omega)$, the variational problem

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega \rho f v dx, \quad v \in H_0^1(\Omega). \tag{17}$$

has a unique solution $u_\rho := R_\rho f \in H_0^1(\Omega)$.

Let A be the bounded operator $A : H_0^1(\Omega) \cap H^2(\Omega) \to L^2(\Omega)$ given by $Au = -\Delta u$, and denote its inverse by R.

We prove

$$\|R\|^{-1}\|R_\rho\| \leq \|\rho\|_{L^\infty(\Omega)} \leq \|A\|\|R_\rho\|, \quad \rho \in L^\infty(\Omega).$$

Fix $\rho_1, \ldots, \rho_N \in L^\infty(\Omega)$. Let $\rho \in V_N = \text{span}\{\rho_1, \ldots \rho_N\}$ and $\widetilde{\rho} \in L^\infty(\Omega)$. As $\rho \to R_\rho$ is linear, we get

$$\|R\|^{-1}\|R_\rho - R_{\widetilde{\rho}}\| \leq \|\rho - \widetilde{\rho}\|_{L^\infty(\Omega)} \leq \|A\|\|R_\rho - R_{\widetilde{\rho}}\|.$$

Define then the distance $\mathbf{d}$ between the resolvent R_ρ and $R_{\widetilde{\rho}}$ as follows

$$\mathbf{d}(R_\rho, R_{\widetilde{\rho}}) = \|\rho - \widetilde{\rho}\|_{L^\infty(\Omega)}.$$

The distance $\mathbf{d}$ yields an appropriate surrogate between resolvents :

$$\mathbf{d}(R_{\widetilde{\rho}}, \mathbf{R}_N) = \text{dist}_{L^\infty(\Omega)}(\widetilde{\rho}, V_N),$$

where $\mathbf{R}_N = \text{span}\{R_{\rho_1}, \ldots, R_{\rho_N}\}$.

Summary

- Surrogates in the multi-dimensional case : this problem is totally open in the multi-dimensional case.
- Elliptic matrices : in dimensions $n \geq 2$ the same problems can be formulated for equations of the form

$$-\partial_i(\sigma_{ij}(x,\mu)\partial_j u) = f.$$

The problem is much more complex in this case since there is no a sole coefficient σ to be identified but rather all the family σ_{ij} with $i, j = 1, ..., n$.

- Elliptic systems : the same problems arise also in the context of elliptic systems such as, for instance, the system of elasticity.

- Evolution equations : the problem addressed make also sense for evolution problems and, in particular, parabolic, hyperbolic and Schrödinger equations.
- Control problems : Greedy and weak greedy methods have been implemented in the context of controllability of finite and infinite-dimensional ODEs. But this has been done for fixed specific data to be controlled. It would be interesting to analyze whether our results can be extended to these controllability problems so to achieve approximations independent of the data to be controlled.