Carleman estimates and applications

NEW TRENDS IN THEORETICAL AND NUMERICAL ANALYSIS OF WAVEGUIDES

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Outline

1. Review in Carleman estimates
2. Carleman estimate for high-order operator at a boundary
3. Application: Inverse problem of the dynamic Schrödinger equation in waveguide
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2. Carleman estimate for high-order operator at a boundary
3. Application: Inverse problem of the dynamic Schrödinger equation in waveguide
Local Carleman estimates

Let $u$ be a solution of

$$P(x, D)u = \sum_{|\alpha| \leq m} a_{\alpha}(x) \partial^{\alpha} u = 0$$

Let $\psi \in C^2$ real valued function, $\nabla \psi \neq 0$.

$(u \equiv 0 \text{ in } \{\psi < 0\}) \Rightarrow (u \equiv 0 \text{ in } \{\psi > 0\})$
Local Carleman estimates

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Holmgren 1901: Analytic coefficients

Erik Holmgren
Local Carleman estimates

- Let $u$ be a solution of
  \[ P(x, D)u = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha u = 0 \]

- Let $\psi \in C^2$ real valued function, $\nabla \psi \neq 0$.

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Petrovsky 1937 : Strictly hyperbolic equations.
Local Carleman estimates

Let $u$ be a solution of

$$P(x, D)u = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha u = 0$$

Let $\psi \in C^2$ real valued function, $\nabla \psi \neq 0$.

$(u \equiv 0 \text{ in } \{\psi < 0\}) \Rightarrow (u \equiv 0 \text{ in } \{\psi > 0\})$

Carleman 1939 : Two independent variables.
Local Carleman estimates

Let $u$ be a solution of

$$P(x, D)u = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha u = 0$$

Let $\psi \in C^2$ real valued function, $\nabla \psi \neq 0$.

$(u \equiv 0 \text{ in } \{\psi < 0\}) \Rightarrow (u \equiv 0 \text{ in } \{\psi > 0\})$

Hörmander, Calderon,...
Local Carleman estimates

Let \( P(x, D) \) be a linear differential operator of order 2 defined in an open set \( \Omega \subset \mathbb{R}^n \). The following type of a priori estimate is called Carleman estimate

\[
\tau \int_{\Omega} (|\nabla u|^2 + \tau^2 |u|^2) e^{2\tau \phi} \, dx \leq C \int_{\Omega} |P(x, D)u|^2 e^{2\tau \phi} \, dx,
\]

for any \( u \in C^2_0(\Omega) \).

- Hörmander, Isakov, Lerner, Robbiano, Zuily...
- If the strong pseudo-convexity condition is satisfied, then we have the Carleman estimate.
Global Carleman estimates

Let $P(x, D)$ be a linear differential operator of order 2 defined in an open set $\Omega \subset \mathbb{R}^n$. The following type of a priori estimate is called global Carleman estimate

$$\tau \int_{\Omega} (|\nabla u|^2 + \tau^2 |u|^2) e^{2\tau \varphi} \leq \int_{\Omega} |P(x, D)u|^2 e^{2\tau \varphi} + \tau \int_{\Gamma_0} |\partial_{\nu} u|^2 e^{2\tau \varphi},$$

for any $u \in H^2(\Omega) \cap H^1_0(\Omega)$. Here $\Gamma_0 \subset \Gamma = \partial \Omega$. 

[The text continues on the next page]
Sufficient Conditions

We are interested in conditions on $\varphi = e^{\beta \psi}$ and $\Gamma_0 \subset \Gamma$ implying the Carleman estimate

$$\tau \int_{\Omega} (|\nabla u|^2 + \tau^2 |u|^2) e^{2\tau \varphi} \leq \int_{\Omega} |P(x, D)u|^2 e^{2\tau \varphi} + \tau \int_{\Gamma_0 \subset \Gamma} |\partial_{\nu} u|^2 e^{2\tau \varphi},$$

for any $u \in H^2(\Omega) \cap H^1_0(\Omega)$.

- $\varphi$ satisfies the strong pseudo-convexity condition in $\Omega$.
- $\varphi$ satisfies the strong Lopatinskii condition on $\Gamma \setminus \Gamma_0$.
- Tataru, Imanuvilov, Isakov, Bellassoued,...
Example 1: Elliptic equation

Let $A \in L^\infty(\Omega, \mathbb{C}^n)$, $q \in L^\infty(\Omega, \mathbb{C})$, $\varphi = e^{\beta \psi}$

$$P(x, D) = -\Delta + A \cdot \nabla + q(x).$$

$$|\nabla \psi(x)| > 0 \quad \forall x \in \Omega, \quad \text{and} \quad \partial_\nu \psi(x) < 0, \quad x \in \Gamma \setminus \Gamma_0 \quad (*)$$

Pseudo-convexity in $\Omega$

Lopatinskii in $\Gamma \setminus \Gamma_0$

Imanuvilov-Fursikov

Let $\Gamma_0 \subset \Gamma$ be an arbitrary open set. Then there exist $\psi \in C^2(\overline{\Omega})$ s.t. $(*)$ is satisfied.
Example 1: Elliptic equation

- Let $A \in L^\infty(\Omega, \mathbb{C}^n)$, $q \in L^\infty(\Omega, \mathbb{C})$

  \[
P(x, D)u = (-\Delta + A \cdot \nabla + q(x))u = f, \quad u \in H^2(\Omega) \cap H^1_0(\Omega),
  \]

- Let $\Gamma_0 \subset \Gamma$ be an arbitrary open set. There exist $C > 0$

  \[
  C \|u\|_{H^1(\Omega)} \leq \|f\|_{L^2(\Omega)} + \|\partial_\nu u\|_{L^2(\Gamma_0)}.
  \]

- Let $\omega \subset \Omega$ be an arbitrary open set. There exist $C > 0$

  \[
  C \|u\|_{H^1(\Omega)} \leq \|f\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)}.
  \]
Exemple 2: Wave equation

Let $\alpha > 0$, $Q = \Omega \times (0, T)$, $Q_\alpha = \Omega \times (\alpha, T - \alpha)$, $\Sigma = \Gamma \times (0, T)$,

$$P(x, D) = \partial^2_t - \Delta + A \cdot \nabla + q(x), \quad \text{(Wave)}$$

Determine $\Gamma_0 \subset \Gamma$ and $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\|u\|_{H^1(Q_\alpha)} \leq \Phi \left( \|f\|_{L^2(Q)} + \|\partial_\nu u\|_{L^2(\Gamma_0 \times (0, T))} \right), \quad \Phi(0) = 0,$$

for any $u \in H^2(Q)$ such that $P(x, D)u = f$ and $u|_{\Sigma} = 0$. 
Exemple 2: Wave Equation

Let $T > 0$, $Q = \Omega \times (0, T)$, $\Sigma = \Gamma \times (0, T)$

$$P(x, D) = \partial_i^2 - \Delta + A \cdot \nabla + q(x)$$

Let $\psi(x) = |x - x_0|^2$, $x_0 \in \mathbb{R}^n \setminus \Omega$

$$\varphi(t, x) = e^{\beta(\psi(x) - \gamma t^2)}, \quad \Gamma_0 \ni \{x \in \Gamma, (x - x_0) \cdot \nu \geq 0\}$$

$T > 2\text{Diam}(\Omega)$ we have

$$\|u\|_{H^1(Q_\alpha)} \leq \|u\|_{H^1(Q)}^{1 - \mu} \left(\|f\|_{L^2(Q)} + \|\partial_\nu u\|_{L^2(\Gamma_0 \times (0, T))}\right)^\mu, \quad \mu \in (0, 1)$$

for any $u \in H^2(Q)$ such that $P(x, D)u = f$ and $u|_\Sigma = 0$. 
Wave equation: with variable coefficients

Let \((\mathcal{M}, g)\) a Riemannian manifold, \(\Delta = \Delta_g\). We assume that there exists a positive and smooth function \(\psi_0\) on \(\mathcal{M}\):

(A.1): \(\psi_0\) is strictly convex on \(\mathcal{M}\) with respect to the metric \(g\):

\[
D^2\psi_0(X, X)(x) > 0, \quad x \in \mathcal{M}, \quad X \in T_x\mathcal{M}\backslash\{0\}.
\]

(A.2): We assume that \(\psi_0(x)\) has no critical points on \(\mathcal{M}\):

\[
\min_{x \in \mathcal{M}} |\nabla \psi_0(x)| > 0.
\]

(A.3): Under (A.1)-(A.2), let \(\Gamma_0 \subset \partial \mathcal{M}\) satisfy

\[
\{x \in \partial \mathcal{M}; \partial^\nu \psi_0 \geq 0\} \subset \Gamma_0.
\]
Wave equation: with variable coefficients

Let us define
\[ \psi(t, x) = \psi_0(x) - \beta (t - t_0)^2 + \beta_0, \quad 0 < \beta < \varrho, \quad 0 < t_0 < T, \quad \beta_0 \geq 0, \]

We define the weight function \( \varphi : \mathcal{M} \times \mathbb{R} \to \mathbb{R} \) by
\[ \varphi(x, t) = e^{\gamma \psi(x, t)}, \]

**Theorem (Bellassoued 04’)**

Assume (A.1), (A.2) and (A.3) then the global Carleman estimates is hold in \((\mathcal{M}, g)\).
Wave equation: with variable coefficients

$$(A.1), (A.2), (A.3) \implies \text{GCC: Bardos-Lebeau-Rauch}$$

Wiffle Ball: $S^2$ deleting $\varepsilon$-neighborhood of the segments along the equator with longitudes $[\pi/6, \pi/2]$, $[5\pi/6, 7\pi/6]$ and $[3\pi/2, 11\pi/6]$
Sufficient Conditions on $\Gamma_0$

**Elliptic operator**

We have a Lipschitz stability estimate of the Cauchy problem if $\Gamma_0 \subset \Gamma$ be an arbitrary open set.

**Hyperbolic operator**

We have a Hölder stability estimate of the Cauchy problem if $\Gamma_0 \subset \Gamma$ is large:

$$\Gamma_0 \supset \{ x \in \partial \Omega; \partial_{\nu} \psi_0 \geq 0 \}$$
Fourier-Bros-Iagolnitzer transform (F.B.I)

- We will specialize to the FBI transform with a Gaussian window:

  \[ \mathcal{F}_\lambda v(s, t) = \alpha \int_{\mathbb{R}} e^{\frac{\lambda}{2} (2i(s-\eta)t-(s-\eta)^2)} v(\eta) d\eta. \]

- We also consider the closely related to Bargmann transform, defined by

  \[ \mathcal{F}_\lambda v(z) = \int_{\mathbb{R}} e^{-\frac{\lambda}{2} (z-\eta)^2} v(\eta) d\eta, \quad z = s - it \]

- \[ i\partial_s (\mathcal{F}_\lambda v(s, t)) = \mathcal{F}_\lambda (\partial_i v)(s, t) \]
Fourier-Bros-Iagolnitzer transform (F.B.I)

\[-(\partial_s^2 + \Delta) (\mathcal{F}_\lambda u(s, t; x)) = \mathcal{F}_\lambda((\partial_t^2 - \Delta)u)(s, t; x) := \mathcal{F}_\lambda(f)(t, s; x)\]

Elliptic eq. \hspace{1cm} Wave eq.

Let $\Gamma_0 \subset \Gamma$ be an arbitrary open set.

By the elliptic Lipschitz stability estimate:

\[\| \mathcal{F}_\lambda u(t, \cdot) \|_{H^1(Q_\alpha)} \leq \left( \| \mathcal{F}_\lambda f(t, \cdot) \|_{L^2(Q)} + \| \partial_n \mathcal{F}_\lambda u(t, \cdot) \|_{L^2(\Gamma_0 \times (0, T))} \right),\]
Fourier-Bros-Iagolnitzer transform (F.B.I)

\[ \| \mathcal{F}_\lambda f(t, .) \|_{L^2(Q)} \leq e^{C\lambda} \| f \|_{L^2(Q)}, \]

\[ \| \partial_\nu \mathcal{F}_\lambda u(t, .) \|_{L^2(\Gamma_0 \times (0,T))} \leq e^{C\lambda} \| \partial_\nu u \|_{L^2(\Gamma_0 \times (0,T))} \]

\[ \| \mathcal{F}_\lambda u(t, .) \|_{H^1(Q_\alpha)} \leq e^{C\lambda} \left( \| f \|_{L^2(Q)} + \| \partial_\nu u \|_{L^2(\Gamma_0 \times (0,T))} \right) \]

Moreover

\[ \| \mathcal{F}_\lambda u(t, .) - u \|_{H^1(Q_\alpha)} \leq \frac{C}{\lambda} \| u \|_{H^2(Q)}. \]

Then we have

\[ \| u \|_{H^1(Q_\alpha)} \leq \frac{C}{\lambda} \| u \|_{H^2(Q)} + e^{C\lambda} \left( \| f \|_{L^2(Q)} + \| \partial_\nu u \|_{L^2(\Gamma_0 \times (0,T))} \right) \]
Fourier-Bros-Iagolnitzer transform (F.B.I)

Let $\Gamma_0 \subset \Gamma$ be an arbitrary open set, we have For the Wave equation:

$$\|u\|_{H^1(Q_\alpha)} \leq \Phi \left( \|f\|_{L^2(Q)} + \|\partial_\nu u\|_{L^2(\Gamma_0 \times (0,T))} \right),$$

$$\Phi(\delta) = C(\log(2 + \delta^{-1}))^{-1}. $$
Outline

1. Review in Carleman estimates

2. Carleman estimate for high-order operator at a boundary

3. Application: Inverse problem of the dynamic Schrödinger equation in waveguide
Setting at a boundary

We consider

- $P$ be a smooth elliptic of order $m = 2\mu$.

\[
P = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha,
\]

with complex-valued coefficients.

- $m/2$ linear smooth boundary operators of order less than $m$

\[
B^k = \sum_{|\alpha| \leq \beta_k} b^k_\alpha(x) \partial^\alpha, \quad k = 1, \ldots, \mu = m/2,
\]

with complex-valued coefficients, defined in some neighborhood of $\partial\Omega$.

Consider the elliptic boundary value problem

\[
\begin{cases}
P u(x) = f(x), & x \in \Omega, \\
B^k u(x) = g^k(x), & x \in \partial\Omega, \quad k = 1, \ldots, \mu.
\end{cases}
\]
Setting at a boundary

Consider the elliptic boundary value problem

\[
\begin{cases}
Pu(x) = f(x), & x \in \Omega, \\
B^k u(x) = g^k(x), & x \in \partial \Omega, \quad k = 1, \ldots, \mu.
\end{cases}
\]

We wish to obtain an estimate of the form

\[
\|e^{\tau \varphi} u\|^2 + |e^{\tau \varphi} T(u)|^2 \lesssim \|e^{\tau \varphi} P(x, D) u\|^2 + \sum_{k=1}^{\mu} |e^{\tau \varphi} B^k(x, D) u|^2,
\]

for \( u \) supported near a point at the boundary \( T(u) \) is the trace of \((u, D_{\nu} u, \ldots, D_{\nu}^{m-1} u)\). If we set

\[
P_{\varphi} = e^{\tau \varphi} P(x, D) e^{-\tau \varphi}; \quad B_{\varphi}^k = e^{\tau \varphi} B^k(x, D) e^{-\tau \varphi}; \quad v = e^{\tau \varphi} u
\]

then the Carleman estimate reads:

\[
\|v\|^2 + |T(v)|^2 \lesssim \|P_{\varphi} v\|^2 + \sum_{k=1}^{\mu} |B_{\varphi}^k v|^2,
\]

Estimates of this form were obtained by Tataru. We give more precise estimates here and include the complex coefficient case.
1st order operator at a boundary

Let $\rho^+ \in \mathbb{C}$ such that $\text{Im}\rho^+ > 0$, $f \in \mathcal{S}(\mathbb{R}^+)$ and $u_0 \in \mathbb{C}$. We consider the boundary problem

$$
\begin{align*}
\left\{ \begin{array}{ll}
(D_{x_n} - \rho^+) u(x_n) = f(x_n) & x_n > 0 \\
u(0) = u_0 \in \mathbb{C}
\end{array} \right. \\
(1^+) \quad (2)
\end{align*}
$$

The first line can be solved by Fourier transformation: Let $f^+$ the extension of $f$ by 0 on $(-\infty, 0)$. Let

$$
v(x_n) = \mathcal{F}^{-1} \left( \frac{1}{\xi_n - \rho^+} \mathcal{F}(f^+) \right)
$$

then $v(x_n)$ the restriction of $v$ to $\mathbb{R}^+$ is a solution of the first line. Since $\text{Im}\rho^+ > 0$ we have

$$
v(x_n) = \mathcal{F}^{-1} \left( \frac{1}{\xi_n - \rho^+} \right) * f^+(x_n)
$$

$$
= i \left( H(x_n) e^{i \rho^+ x_n} \right) * f^+(x_n) = i \int_0^{x_n} e^{i \rho^+(x_n-y_n)} f(y_n) dy_n
$$
1st order operator at a boundary

Let $\rho^+ \in \mathbb{C}$ such that $\text{Im} \rho^+ > 0$, $f \in \mathcal{L}(\mathbb{R}^+)$ and $u_0 \in \mathbb{C}$. We consider the boundary problem

$$\begin{cases}
(D_{x_n} - \rho^+) u(x_n) = f(x_n) & x_n > 0 \\
u(0) = u_0 \in \mathbb{C}
\end{cases} \quad (1^+)$$

The full problem is then solved by reduction to a semihomogeneous problem: Set $w(x_n) = u(x_n) - v(x_n)$ then since $v(0) = 0$, $w(x_n)$ must solve

$$\begin{cases}
(D_{x_n} - \rho^+) w(x_n) = 0 & x_n > 0 \\
w(0) = u_0 \in \mathbb{C}
\end{cases}$$

that is

$$w(x_n) = e^{i\rho^+ x_n} u_0$$

So the solution of the full problem is

$$u(x_n) = \underbrace{e^{i\rho^+ x_n} u_0}_{\in \mathcal{L}(\mathbb{R}^+)} + i \underbrace{\int_0^{x_n} e^{i\rho^+ (x_n-y_n)} f(y_n) dy_n}_{\in \mathcal{L}(\mathbb{R}^+)}$$
1st order operator at a boundary

Let $\rho^+ \in \mathbb{C}$ such that $\text{Im}(\rho^+) > 0$, $f \in \mathcal{S}(\mathbb{R}^+)$ and $u_0 \in \mathbb{C}$. We consider the boundary problem

$$
\begin{align*}
\begin{cases} 
(D_{x_n} - \rho^+) u(x_n) = f(x_n) & x_n > 0 \\
u(0) = u_0 \in C
\end{cases} \\
(1^+) \\
(2)
\end{align*}
$$

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$$
\begin{align*}
\begin{cases} 
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w(0) = u_0 \in C
\end{cases}
\end{align*}
$$

that is

$$w(x_n) = e^{i\rho^+ x_n} u_0$$

So the solution of the full problem is

$$u(x_n) = e^{i\rho^+ x_n} u_0 + i \int_0^{x_n} e^{i\rho^+(x_n-y_n)} f(y_n) dy_n$$

$\in \mathcal{S}(\mathbb{R}^+)$
1st order operator at a boundary

Let $\rho^- \in \mathbb{C}$ such that $\text{Im}\rho^- < 0$, $f \in \mathcal{S}(\mathbb{R}^+)$ and $u_0 \in \mathbb{C}$. We consider the boundary problem

$$(D_{x_n} - \rho^-) u(x_n) = f(x_n) \quad x_n > 0 \quad (1^-)$$

One solution of (1^-) is found by taking the restriction to $(0, \infty)$ of the $L^2$ function:

$$v(x_n) = \mathcal{F}^{-1} \left( \frac{1}{\xi_n - \rho^-} \right) \ast f^+(x_n)$$

$$= \left( -iH(-x_n)e^{i\rho^-x_n} \right) \ast f^+(x_n) = -i \int_{x_n}^{\infty} e^{i\rho^-(x_n-y_n)}f(y_n)dy_n$$

it is locally absolutely continous. Now $v(x_n)$ is the only solution belonging to $L^2(\mathbb{R}^+)$: for any other solution $u(x_n)$ let

$$w(x_n) = u(x_n) - v(x_n)$$

then:

$$(D_{x_n} - \rho^-) w(x_n) = 0 \quad x_n > 0$$

which is of the form $w(x_n) = e^{i\rho^-x_n}c_0$, that is not in $L^2(\mathbb{R}^+)$ for $c_0 \neq 0$ since $\text{Im}\rho^- < 0$. 

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Carleman estimates

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1st order operator at a boundary

We give special names to the occurring operators:

\[ \lambda^+(\xi_n) = \frac{1}{\xi_n - \rho^+} \quad \text{a symbol in } S^{-1}(\mathbb{R} \times \mathbb{R}) \]

\[ \Lambda^+(D_n)f = \mathcal{F}^{-1} \left( \lambda^+(\xi_n) \mathcal{F}(f) \right) \quad \text{the corresponding } \Psi d.o \]

\[ \Lambda^+_{\Omega}(D_n)f = r^+ \mathcal{F}^{-1} \left( \lambda^+(\xi_n) \mathcal{F}(f^+) \right) \quad \text{its restriction to } \Omega = \mathbb{R}^+ \]

Let us also introduce the multiplication operator

\[ k(x_n, \rho^+ )c = H(x_n)e^{i\rho^+ x_n}c, \quad c \in \mathbb{C}. \]

Then the conclusions that the operator:

\[ A_+ : \mathcal{S}(\mathbb{R}^+) \to \mathcal{S}(\mathbb{R}^+) \times \mathbb{C} \]

\[ u \quad \mapsto \quad (f = (D_n - \rho^+)u, u(0)) \]

is bijective and has the inverse operator

\[ A_+^{-1} : \mathcal{S}(\mathbb{R}^+) \times \mathbb{C} \to \mathcal{S}(\mathbb{R}^+) \]

\[ (f, u_0) \quad \mapsto \quad k^+(x_n)u_0 + \Lambda^+_{\Omega}(D_n)f \]
1st order operator at a boundary

\[ \lambda^-(\xi_n) = \frac{1}{\xi_n - \rho^-} \text{ a symbol in } S^{-1}(\mathbb{R} \times \mathbb{R}) \]

\[ \Lambda^-(D_n)f = \mathcal{F}^{-1} \left( \lambda^-(\xi_n)\mathcal{F}(f) \right) \text{ the corresponding } \Psi d.o \]

\[ \Lambda_\Omega^-(D_n)f = r^+ \mathcal{F}^{-1} \left( \lambda^-(\xi_n)\mathcal{F}(f^+) \right) \text{ its restriction to } \Omega = \mathbb{R}^+ \]

Then the conclusions from the example with \( \rho^- \) is that the operator:

\[ A_- : \mathcal{I}(\mathbb{R}^+) \rightarrow \mathcal{I}(\mathbb{R}^+) \]

\[ u \mapsto f = (D_n - \rho^-)u \]

is bijective and has the inverse operator

\[ A_-^{-1} : \mathcal{I}(\mathbb{R}^+) \rightarrow \mathcal{I}(\mathbb{R}^+) \]

\[ f \mapsto \Lambda_\Omega^-(D_n)f \]
2nd order operator at a boundary

The general theory for elliptic boundary value problems can be regarded as built up from the laste two examples. Consider the Dirichlet problem on the half space $\mathbb{R}^n_+$

\[
\begin{cases}
(-\Delta + 1) u(x) = f(x) & x_n > 0 \\
u(x', 0) = u_0(x') & x_n = 0
\end{cases}
\]

By Fourier transform in the $x'$-varibale

\[
\begin{cases}
\left(D_{x_n}^2 + \langle \xi' \rangle^2 \right) \hat{u}(\xi', x_n) = \hat{f}(\xi', x_n) & x_n > 0 \\
\hat{u}(\xi', 0) = \hat{u}_0(\xi') & x_n = 0
\end{cases}
\]

Now

\[
\left(D_{x_n}^2 + \langle \xi' \rangle^2 \right) = \left(D_{x_n} - i \langle \xi' \rangle \right) \left(D_{x_n} + i \langle \xi' \rangle \right) = (D_{x_n} - \rho^+) (D_{x_n} - \rho^-)
\]

We can solve (1) by breaking it up in two problems

\[
\begin{cases}
(D_{x_n} - \rho^+) \hat{u}(x_n) = v & x_n > 0 \\
\hat{u}(0) = \hat{u}_0(\xi') & x_n = 0 ;
\end{cases} \quad \quad \quad \quad \begin{cases}
(D_{x_n} - \rho^-) v = \hat{f} & x_n > 0
\end{cases}
\]
2nd order operator at a boundary

We can solve (1) by breaking it up in two problems

\[
\begin{align*}
\begin{cases}
(D_{x_n} - \rho^+) \hat{u}(x_n) &= v \quad x_n > 0 \\
\hat{u}(0) &= \hat{u}_0(\xi') \quad x_n = 0
\end{cases}
\end{align*}
\]

\[
(D_{x_n} - \rho^-) v = \hat{f} \quad x_n > 0
\]

\[
v = A_-^{-1} \hat{f}(\xi', x_n) = \Lambda_{\Omega}^- (D_n) \hat{f}(\xi', x_n), \quad \rho^- = -i \langle \xi' \rangle
\]

\[
\hat{u}(\xi', x_n) = A_+^{-1} (v(\xi', \cdot), \hat{u}_0(\xi')) = k^+ (x_n) \hat{u}_0(\xi')) + \Lambda_{\Omega}^+ (D_n) v, \quad \rho^+ = i \langle \xi' \rangle
\]

\[
\hat{u}(\xi', x_n) = A_+^{-1} (v(\xi', \cdot), \hat{u}_0(\xi'))
\]

\[
= k^+ (x_n) \hat{u}_0(\xi') + \Lambda_{\Omega}^+ (D_n) \Lambda_{\Omega}^- (D_n) \hat{f}(\xi', x_n)
\]
2nd order operator at a boundary

We can solve (1) by breaking it up in two problems

\[
\begin{aligned}
(D_x - \rho^+) \hat{u}(x) &= v & x > 0 \\
\hat{u}(0) &= \hat{u}_0(\xi') & x = 0 \quad ; \quad (D_x - \rho^-) v = \hat{f} & x > 0
\end{aligned}
\]

\[v = A^{-1}_-(D_x)f(\xi', x) = \Lambda^-_\Omega(D_x)f(\xi', x), \quad \rho^- = -i \langle \xi' \rangle\]

\[\hat{u}(\xi', x) = A^{-1}_+(v(\xi', \cdot), \hat{u}_0(\xi')) = k^+(x)\hat{u}_0(\xi') + \Lambda^+_\Omega(D_x)v, \quad \rho^+ = i \langle \xi' \rangle\]

\[\hat{u}(\xi', x) = A^{-1}_+(v(\xi', \cdot), \hat{u}_0(\xi')) = k^+(x)\hat{u}_0(\xi') + \Lambda^+_\Omega(D_x)\Lambda^-_\Omega(D_x)f(\xi', x)\]
2nd order operator at a boundary

We can solve (1) by breaking it up in two problems

\[
\begin{cases}
(D_{x_n} - \rho^+) \hat{u}(x_n) = v \quad x_n > 0 \\
\hat{u}(0) = \hat{u}_0(\xi') \quad x_n = 0
\end{cases}
\] ; \quad
(D_{x_n} - \rho^-) v = \hat{f} \quad x_n > 0

\[
v = A^{-1}_-(\xi', x_n) = \Lambda^{-}_\Omega(D_n)\hat{f}(\xi', x_n), \quad \rho^- = -i \langle \xi' \rangle
\]

\[
\hat{u}(\xi', x_n) = A^{-1}_+(v(\xi', \cdot), \hat{u}_0(\xi')) = k^+(x_n)\hat{u}_0(\xi') + \Lambda^+\Omega(D_n)v, \quad \rho^+ = i \langle \xi' \rangle
\]

\[
\hat{u}(\xi', x_n) = A^{-1}_+(v(\xi', \cdot), \hat{u}_0(\xi')) = k^+(x_n)\hat{u}_0(\xi') + \Lambda^+\Omega(D_n)\Lambda^{-}_\Omega(D_n)\hat{f}(\xi', x_n)
\]
2nd order operator at a boundary

2nd-order operators at the boundary were precisely treated by G. Lebeau and L. Robbiano (95, 97).

Set

\[ P = -\Delta = \sum_j D_j^2; \quad \partial \Omega = \{ x_n = 0 \}, \quad \Omega = \{ x_n > 0 \} \]

We write \( x = (x', x_n) \) and \( \xi = (\xi', \xi_n) \).

Take \( \varphi = \varphi(x_n) \) such that \( \varphi' > 0 \). Then

\[
P_\varphi = \left( D_{x_n} + i\tau \varphi'(x) \right)^2 + \sum_{1 \leq j \leq n-1} D_j^2
\]

\[
= \left( D_{x_n} + i(\tau \varphi'(x) + |D'|) \right) \left( D_{x_n} + i(\tau \varphi'(x) - |D'|) \right),
\]

where \( |D'| = \text{Op}(|\xi'|) \).

Consider the principal symbol

\[
p_\varphi(x, \xi) = p(x, \xi + i\tau \varphi') = \left( \xi_n + i(\tau \varphi'(x) + |\xi'|) \right) \left( \xi_n + i(\tau \varphi'(x) - |\xi'|) \right)
\]
2nd order operator at a boundary

Principal symbol

\[ p_\varphi(x, \xi) = p(x, \xi + i\tau \varphi') = (\xi_n + i(\tau \varphi'(x) + |\xi'|))(\xi_n + i(\tau \varphi'(x) - |\xi|)) = (\xi_n - \rho_1)(\xi_n - \rho_2) \]

In the low-frequency regime, \(|\xi'\)| small,

We have (a microlocal perfect elliptic estimate)

\[ \|v\|_{2,\tau} + |T(v)|_{1,1/2,\tau} \lesssim \|P_\varphi v\|_{L^2} \quad (+ \cdots) \]
2nd order operator at a boundary

Principal symbol

\[ p_\varphi(x, \xi) = p(x, \xi + i\tau \varphi') = (\xi_n + i(\tau \varphi'(x) + |\xi'|))(\xi_n + i(\tau \varphi'(x) - |\xi|)) = (\xi_n - \rho_1)(\xi_n - \rho_2) \]

In the high-frequency regime, \( |\xi'| \) large,

We have

\[ \tau^{-1/2} \|v\|_{2,\tau} + |T(v)|_{1,1/2,\tau} \lesssim \|P_\varphi v\|_{L^2} + \text{boundary norm} \quad (+ \cdots) \]

The boundary norm can be of Dirichlet, Neumann, Robin type...
High-order operator at a boundary

Set $\varrho' = (x, \xi', \tau)$

We have

$$p_\varphi(\varrho', \xi_n) = \prod_{j=1}^{m} (\xi_n - \rho_j(\varrho')) = p_\varphi^+(\varrho', \xi_n)p_\varphi^-(\varrho', \xi_n)p_\varphi^0(\varrho', \xi_n),$$

with

$$p_\varphi^\pm(\varrho', \xi_n) = \prod_{\pm \Im \rho_j > 0} (\xi_n - \rho_j), \quad p_\varphi^0(\varrho', \xi_n) = \prod_{\Im \rho_j = 0} (\xi_n - \rho_j).$$

$p_\varphi^-$ yields a prefect elliptic estimate.

We set

$$\kappa_\varphi(\varrho', \xi_n) = p_\varphi^+(\varrho', \xi_n)p_\varphi^0(\varrho', \xi_n)$$
High-order operator at a boundary: $\kappa_\varphi (\varphi', \xi_n) = 1$

Principal symbol

$$p_\varphi (x, \xi) = p(x, \xi + i\tau \varphi') = \prod_{j=1}^{k} (\xi_n - \rho_j(x, \tau, \xi'))$$

If all the roots have a negative imaginary part,

Write $p_\varphi = a + ib$, $a$ and $b$ both self adjoint and $A = a(x, D, \tau)$, $B = b(x, D, \tau)$, we have

$$\| P_\varphi (x, D)v \|_{L^2}^2 = \| Av \|_{L^2}^2 + \| Bv \|_{L^2}^2 + 2 \Re (Av, iBv)$$
Bézout Matrices

Given two univariate polynomials \( a(\zeta) = \sum_{j=0}^{m} a_j \zeta^j \), \( b(\zeta) = \sum_{j=0}^{m} b_j \zeta^j \), we build the following bivariate polynomial

\[
B_{a,b}(\zeta, \tilde{\zeta}) = \frac{a(\zeta)b(\tilde{\zeta}) - a(\tilde{\zeta})b(\zeta)}{\zeta - \tilde{\zeta}} = \sum_{j,k=0}^{m-1} g_{j,k} \zeta^j \tilde{\zeta}^k,
\]

called the Bézoutian of \( a \) and \( b \), and the corresponding symmetric matrix \( g_{a,b} = (g_{j,k}) \) of size \( m \times m \) with entries \( g_{j,k} \), bilinear in the coefficients of \( a \) and \( b \), is called the Bézout matrix and given by:

\[
g_{j,k} = \sum_{\ell=0}^{\min(j,k)} \left( b_{\ell} a_{j+k-\ell+1} - b_{j+k-\ell+1} a_{\ell} \right),
\]

upon letting \( a_k = b_k := 0 \) for \( k > m \) and \( k < 0 \). With this Bézout matrix we associate the following bilinear form:

\[
\widetilde{B}_{a,b}(z, z') = \sum_{j,k=0}^{m-1} g_{j,k} z_j \bar{z}_k', \quad z = (z_0, \ldots, z_{m-1}), \ z' = (z'_0, \ldots, z'_{m-1}) \in \mathbb{C}^m.
\]
Hermite Theorem

The following Hermite Theorem providing a relation between the roots of a polynomial and the Bézout matrix associated with the real and imaginary parts of the polynomial.

Lemma (Hermite Theorem)

Let \( h(\zeta) = a(\zeta) + ib(\zeta) \) be a polynomial of degree \( k \geq 1 \), where \( a(\zeta) \) and \( b(\zeta) \) are polynomials with real coefficients. Assume that all the roots of \( h(\zeta) \) are in the lower complex half-plane \( \{ \Im \zeta < 0 \} \). Then the roots of \( a(\zeta) \) and \( b(\zeta) \) are real and distinct. Moreover, the bilinear form \( \tilde{B}_{a,b}(z, z') \) is positive, that is there exists \( C > 0 \) such that

\[
\tilde{B}_{a,b}(z, z) \geq C |z|^2, \quad z \in \mathbb{C}^k.
\]
A generalized Green formula

Consider two smooth and real symbols \( a(x, \xi, \tau) \) and \( b(x, \xi, \tau) \). The following identity holds true

\[
2 \text{Re} (A v, i B v) = H_{a,b}(v) + \mathcal{B}_{a,b}(v) + R(v), \quad A = a(x, D, \tau), \ B = b(x, D, \tau),
\]

for any \( v \in \mathcal{L}(\mathbb{R}^n_{+}) \). Here:

- \( \mathcal{B}_{a,b} \) is the boundary quadratic form with symbol the Bézout matrix \( \tilde{B}_{a,b}(z, z') \).
- \( H_{a,b} \) is an interior quadratic form with real symbol

\[
h_{a,b}(\varrho) = \text{sub}(a, b)(\varrho) = \{a, b\} + \sum_{|\alpha|=1} (b \partial_\xi^\alpha \partial_x^\alpha a - a \partial_\xi^\alpha \partial_x^\alpha b) .
\]

- the remainder term \( R(v) \) is a quadratic form that satisfies

\[
|R(v)| \leq C \|v\|_{k,-1,\tau}^2.
\]
High-order operator at a boundary: $\kappa_\varphi(\xi_n) = 1$

Principal symbol

$$p_\varphi(x, \xi) = p(x, \xi + i\tau \varphi') = \prod_{j=1}^{k} (\xi_n - \rho_j(x, \tau, \xi'))$$

If all the roots have a negative imaginary part,

we want to prove (a microlocal perfect elliptic estimate)

$$\|v\|_{m,\tau} + |T(v)|_{k-1,1/2,\tau} \lesssim \|p_\varphi v\|_{L^2} ( + \cdots )$$
Ideas of the proof

We have

\[ 2\Re(Av, iBv) = H_{a,b}(v) + \mathcal{B}_{a,b}(v) + R(v), \quad A = a(x, D, \tau), \quad B = b(x, D, \tau), \]

Write \( p_\varphi = a + ib \), \( a \) and \( b \) both self adjoint.

\[ \|A(x, D, \tau)v\|_{L^2}^2 + \|B(x, D, \tau)v\|_{L^2}^2 \geq C\|v\|_{k,\tau}^2 - C'|T(v)|_{k-1,1/2,\tau}^2 \]

(the roots of \( a \) and \( b \) are real and distinct)

With a generalized green formula we have

\[ 2\Re(Av, iBv)_{L^2} \geq H_{a,b}(v) + \mathcal{B}_{a,b}(v) - C\|v\|_{k,-1/2,\tau}^2 \]

Sub-ellipticity property:

\[ p_\varphi(\varrho, \xi_n) = 0 \quad \Rightarrow \quad \{a, b\}(\varrho, \xi_n) > 0 \quad \Rightarrow \quad h_{a,b}(\varrho, \xi_n) > 0. \]

The position of the roots and the Hermite Theorem give

\[ \mathcal{B}_{a,b}(v) \gtrsim |T(v)|_{k-1,1/2,\tau}^2 \]
High-order operator at a boundary

Set $\varrho' = (x, \xi', \tau)$

We have

$$p_\varphi(\varrho', \xi_n) = \prod_{j=1}^{m} (\xi_n - \rho_j(\varrho')) = p^+_\varphi(\varrho', \xi_n)p^-_\varphi(\varrho', \xi_n)p^0_\varphi(\varrho', \xi_n),$$

with

$$p^\pm_\varphi(\varrho', \xi_n) = \prod_{\pm \text{Im}\rho_j > 0} (\xi_n - \rho_j), \quad p^0_\varphi(\varrho', \xi_n) = \prod_{\text{Im}\rho_j = 0} (\xi_n - \rho_j).$$

$p^-_\varphi$ yields a prefect elliptic estimate.

We set

$$\kappa_\varphi(\varrho', \xi_n) = p^+_\varphi(\varrho', \xi_n)p^0_\varphi(\varrho', \xi_n)$$
High-order operator at a boundary

Boundary operators: $B^k, \quad k = 1, \ldots, \mu$

Conjugated operators: $B^k_\varphi = e^{\tau \varphi} B^k e^{-\tau \varphi}$

Principal symbol: $b^k_\varphi (\xi', \xi_n) \equiv b^k_\varphi (\xi_n)$

Strong Lopatinski condition:
The set $\{b^k_\varphi (\xi_n)\}_{k=1, \ldots, \mu}$ is complete modulo $\kappa_\varphi (\xi_n)$ as polynomials in $\xi_n$.

For all $f(\xi_n)$ polynomial, there exist $c_1, \ldots, c_\mu \in \mathbb{C}$ and $q(\xi_n)$ polynomial such that

$$f(\xi_n) = \sum_{k=1}^{\mu} c_k b^k_\varphi (\xi_n) + q(\xi_n) \kappa_\varphi (\xi_n)$$
High-order operator at a boundary

We have thus obtained

**Theorem (Bellassoued, Le Rousseau)**

*Under*

- **sub-ellipticity condition,**
- **strong Lopatinskiii condition,**

*Let* $x_0 \in \partial \Omega$. **There exist** $W$ **a nbhd of** $x_0$, $C > 0$, **and** $\tau_0 > 0$ **such that at the boundary**

$$
\tau^{-1} \| e^{\tau \varphi} u \|^2_{m, \tau} + |e^{\tau \varphi} T(u)|^2_{m-1, 1/2, \tau} \\
\leq C \left( \| e^{\tau \varphi} P(x, D) u \|^2_{L^2} + \sum_{k=1}^{\mu} |e^{\tau \varphi} B^k(x, D) u|^2_{m-1/2 - \beta_k, \tau} \right),
$$

*for* $\tau \geq \tau_0$ **and** $u = w|\Omega$ **with** $w \in \mathcal{C}_0^\infty(W)$.
Outline

1. Review in Carleman estimates
2. Carleman estimate for high-order operator at a boundary
3. Application: Inverse problem of the dynamic Schrödinger equation in waveguide
Dynamic Schrödinger equation in waveguide

Let \( \omega \) is an open connected bounded domain in \( \mathbb{R}^{n-1} \), \( n \geq 3 \), with boundary \( \partial \omega \), and we consider \( \Omega := \omega \times \mathbb{R} \), in \( \mathbb{R}^n \), with cross section \( \omega \). Its boundary is denoted by \( \Gamma := \partial \omega \times \mathbb{R} \). Given \( T > 0 \), \( p : \Omega \to \mathbb{R} \) and \( u_0 : \Omega \to \mathbb{R} \), we consider the Schrödinger equation,

\[-i \partial_t u(x, t) - \Delta u(x, t) + p(x)u(x, t) = 0, \quad (x, t) \in \Omega \times (0, T),\]

associated with the initial data \( u_0 \),

\[ u(x, 0) = u_0(x), \quad x \in \Omega, \]

and the homogeneous Dirichlet boundary condition,

\[ u(x, t) = 0, \quad (x, t) \in \Gamma \times (0, T). \]
Dynamic Schrödinger equation in waveguide

Given an arbitrary relatively open subset $S_* \subset \partial \omega$, we aim for determining the unknown potential $p = p(x)$ from one Neumann observation of the function $u_p$ on $\Sigma_* := \Gamma_* \times (0, T)$, where $\Gamma_* := S_* \times \mathbb{R}$ is an infinitely extended strip.

The uniqueness issue: is to know whether any two admissible potentials $p_j, j = 1, 2$, are equal, i.e. $p_1(x) = p_2(x)$ for a.e. $x \in \Omega$, if their observation data coincide, that is, if we have

$$\partial_\nu u_{p_1}(x, t) = \partial_\nu u_{p_2}(x, t), \ (x, t) \in \Sigma_*.$$
Dynamic Schrödinger equation in waveguide

**Theorem (Bellassoued-Kian-Soccorsi)**

Assume that

$$\exists \kappa > 0, \exists d_0, |u_0(x', x_n)| \geq \kappa \langle x_n \rangle^{-d_0/2}, (x', x_n) \in \Omega.$$  

For $p_j \in \mathcal{P}_{\text{admissible}}(p_0, \omega_0), j = 1, 2$, we denote by $u_j$ the solution to the IBVP, where $p_j$ is substituted for $p$. Then, for any $\epsilon \in (0, 1)$, there exists a constant $C > 0$, such that we have

$$\|p_1 - p_2\|_{L^2(\Omega)} \leq C \left( \|\partial^\nu(u_1 - u_2)\|_\ast + |\log \|\partial^\nu(u_1 - u_2)\|_\ast|^{-1} \right)^\epsilon.$$

$$\|\partial^\nu u\|_\ast := \|\partial^\nu u\|_{H^1(0,T;L^2(\Gamma_\ast))}, \ u \in \mathcal{H}^2.$$
Weak observability for the Schrödinger eq.

Let the linear Schrödinger equation

\[
\begin{aligned}
&i \partial_t v + \Delta v = 0 & \text{in } \Omega \times [0, \infty), \\
v(x, t) = 0 & \text{on } \partial \Omega \times [0, \infty), \\
v(x, 0) = v_0(x) & \text{in } \Omega
\end{aligned}
\]

Theorem

Let $\Gamma_0$ be a non-empty open subset of $\partial \Omega$. For any $\mu \in (0, 1)$ and $T > 0$, there exists $C > 0$ and $\lambda_0 > 0$ such that for any non-identically zero initial data $u_0 \in H^2(\Omega)$, we have

\[
\|v_0\|_{L^2(\Omega)}^2 \leq C \left[ \frac{1}{\gamma^{2\mu}} \|v_0\|_{H^2(\Omega)}^2 + e^{C\gamma} \int_0^T \int_{\Gamma_0} |\partial_\nu v(x, t)|^2 \, dx \, dt \right]
\]

for any $\gamma \geq \gamma_0$. 

Weak observability for the Schrödinger eq.

Let the linear Schrödinger equation

\[
\begin{cases}
    i\partial_t v + \Delta v = 0 & \text{in } \Omega \times [0, \infty), \\
    v(x, t) = 0 & \text{on } \partial \Omega \times [0, \infty), \\
    v(x, 0) = v_0(x) & \text{in } \Omega
\end{cases}
\]

(1)

Theorem

Let \( \Gamma_0 \) be a non-empty open subset of \( \partial \Omega \). For any \( \mu \in (0, 1) \) and \( T > 0 \), there exists \( C > 0 \) and \( \lambda_0 > 0 \) such that for any non-identically zero initial data \( u_0 \in H^2(\Omega) \), we have

\[
\| v_0 \|_{L^2(\Omega)}^2 \leq C \left[ \frac{1}{\gamma^{2\mu}} \| v_0 \|_{H^2(\Omega)}^2 + e^{C\gamma} \int_0^T \int_{\Gamma_0} |\partial_{\nu} v(x, t)|^2 \, dx \, dt \right]
\]

for any \( \gamma \geq \gamma_0 \).
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Weak observability for the Schrödinger eq.

Step 1: Observability of the Heat eq.
Let $h > 0$, we consider the linear heat equation

$$\begin{cases} 
\partial_s w + \Delta w = f & \text{in } \Omega \times (0, h), \\
w(x, s) = 0 & \text{on } \partial \Omega \times (0, h), 
\end{cases} \tag{2}$$

Theorem

Let $\Gamma_0 \subset \partial \Omega$. Then there exists $C_h > 0$ such that

$$\int_{\Omega} |w(x, 0)|^2 \, dx \leq C_h \left( \int_0^h \int_{\Gamma_0} |w(x, s)|^2 \, dx \, ds + \int_0^h \int_{\Omega} |f(x, s)|^2 \, dx \, ds \right)$$

Carleman parabolic estimates.
Weak observability for the Schrödinger eq.

Step 1: Observability of the Heat eq.
Let $h > 0$, we consider the linear heat equation

$$\begin{cases}
\partial_sw + \Delta w = f & \text{in } \Omega \times (0, h), \\
w(x, s) = 0 & \text{on } \partial\Omega \times (0, h),
\end{cases}$$

(2)

Theorem

Let $\Gamma_0 \subset \partial\Omega$. Then there exists $C_h > 0$ such that

$$\int_\Omega |w(x, 0)|^2 \, dx \leq C_h \left( \int_0^h \int_{\Gamma_0} |w(x, s)|^2 \, dx \, ds + \int_0^h \int_\Omega |f(x, s)|^2 \, dx \, ds \right)$$

Carleman parabolic estimates.
Weak observability for the Schrödinger eq.

Step 2: Connection between the Schrödinger’s and the heat eqs.
Let \( \mu \in (0, 1) \) and choose \( m \in \mathbb{N}^* \) such that \( 0 < \mu + \frac{1}{2m} < 1 \). Put \( \rho = 1 - \frac{1}{2m} > \mu \). For any \( \gamma \geq 1 \), the function

\[
F_\gamma(z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iz\tau} e^{-(\tau/\gamma\rho)^{2m}} d\tau, \quad z \in \mathbb{C},
\]

is holomorphic in \( \mathbb{C} \). Moreover, there exists four positive constants \( C_1, C_2, C_3 \) and \( C_4 \) (independent on \( \gamma \)) such that

\[
|F_\gamma(z)| \leq C_1 \gamma^\rho e^{C_2 \gamma |\text{Im}z|^1/\rho}, \quad \forall z \in \mathbb{C},
\]

and

\[
|F_\gamma(z)| \leq C_1 \gamma^\rho e^{-C_3 \gamma |\text{Re}z|^{1/\rho}} \quad \forall z \in \{z \in \mathbb{C}, |\text{Im}z| \leq C_4 |\text{Re}z|\}.
\]
Weak observability for the Schrödinger eq.

Step 2: Connection between the Schrödinger’s and the heat eqs.

Now, let \( s, \ t \in \mathbb{R} \), we introduce the following Fourier-Bros-Iagolnitzer transformation as in (Lebeau-Robbiano)

\[
    w_{\gamma,t}(x, s) = \int_{\mathbb{R}} F_{\gamma}(t + is - \tau) \varphi(\tau) w(x, \tau) d\tau,
\]

where \( \varphi \in C_0^\infty(\mathbb{R}) \), \( t \in I \) and \( s \in (0, h) \).

Let \( \nu \) be a solution of the following boundary value problem in

\[
\begin{cases}
    (i\partial_t + \Delta)\nu(x, t) = 0 & \text{in } \Omega \times (0, T), \\
    \nu = 0 & \text{on } \partial\Omega \times (0, T).
\end{cases}
\]

In connection with the operator \((i\partial_t + \Delta)\), we define the parabolic operator in \( \Omega \times (0, h) \) for some \( h > 0 \) by \((\partial_s + \Delta)\).

Since

\[
    (\partial_s + \Delta)\nu_{\gamma,t}(x, s) = \int_{\mathbb{R}} F_{\gamma}(t + is - \tau)(i\partial_{\tau} + \Delta)(\varphi(\tau)\nu(x, \tau)) d\tau
\]
We have \( v_{\gamma,t} \) satisfies the following IBVP in \( \Omega \times (0, h) \).

\[
\begin{cases}
(\partial_s + \Delta) v_{\gamma,t}(x, s) = G_{\gamma,t}(x, s) & \text{in } \Omega \times (0, h), \\
v_{\gamma,t}(x, s) = 0 & \text{on } \partial\Omega \times (0, h), \\
v_{\gamma,t}(x, 0) = (F_{\gamma} \ast \varphi v(x, \cdot))(t) & \text{in } \Omega,
\end{cases}
\]

where

\[
G_{\gamma,t}(x, s) = -i \int_{\mathbb{R}} F_{\gamma}(r + is - \tau) \varphi'(\tau) u(x, \tau) d\tau.
\]
Step 3: Estimations

1. There exists \( \varphi \in C_0^\infty(\mathbb{R}) \) and \( I \subset (0, T) \) such that

\[
\| G_{\gamma,t} \|_{L^2(\Omega \times (0,h))} \leq Ce^{-C\gamma} \| v \|_{L^2(\Omega \times (0,T))}, \quad t \in I.
\]

2. \[
\| v(x, \cdot) \|_{L^2(I)}^2 \leq \| \hat{\varphi}v(x, \cdot) - \hat{F}_\gamma \hat{\varphi}v(x, \cdot) \|_{L^2(\mathbb{R})}^2 + \int_I |v_{\gamma,t}(x, 0)|^2 \, dt \\
\leq \frac{C}{\gamma^{2\mu}} \| \partial_t(\varphi v)(x, \cdot) \|_{L^2(\mathbb{R})}^2 + \int_I |v_{\gamma,t}(x, 0)|^2 \, dt
\]
Thank you