Carleman estimates and applications

NEW TRENDS IN THEORETICAL AND NUMERICAL ANALYSIS OF WAVEGUIDES

Mourad Bellassoued

University of Tunis El Manar, Tunisia

Porquerolles, 16-19 Mai 2016

Outline

Outline

Outline

1. Review in Carleman estimates
2. Carleman estimate for high-order operator at a boundary
3. Application: Inverse problem of the dynamic Schrödinger equation in waveguide

Outline

Local Carleman estimates

- Let u be a solution of

$$P(x,D)u = \sum_{|\alpha| \leqslant m} a_\alpha(x)\partial^\alpha u = 0$$

- Let $\psi \in C^2$ real valued function, $\nabla\psi \neq 0$.
- $(u \equiv 0 \quad in\, \{\psi < 0\}) \Rightarrow (u \equiv 0 \quad in\, \{\psi > 0\})$

Local Carleman estimates

- Let u be a solution of

$$P(x,D)u = \sum_{|\alpha|\leqslant m} a_\alpha(x)\partial^\alpha u = 0$$

- Let $\psi \in C^2$ real valued function, $\nabla\psi \neq 0$.
- $(u \equiv 0 \quad in\,\{\psi < 0\}) \Rightarrow (u \equiv 0 \quad in\,\{\psi > 0\})$

Holmgren 1901: Analytic coefficients

Erik Holmgren

Local Carleman estimates

- Let u be a solution of

$$P(x, D)u = \sum_{|\alpha| \leqslant m} a_\alpha(x)\partial^\alpha u = 0$$

- Let $\psi \in C^2$ real valued function, $\nabla\psi \neq 0$.
- $(u \equiv 0 \quad in\, \{\psi < 0\}) \Rightarrow (u \equiv 0 \quad in\, \{\psi > 0\})$

Petrovsky 1937 : Strictly hyperbolic equations.

Local Carleman estimates

- Let u be a solution of

$$P(x,D)u = \sum_{|\alpha|\leqslant m} a_\alpha(x)\partial^\alpha u = 0$$

- Let $\psi \in C^2$ real valued function, $\nabla\psi \neq 0$.
- $(u \equiv 0 \quad in\, \{\psi < 0\}) \Rightarrow (u \equiv 0 \quad in\, \{\psi > 0\})$

Carleman 1939 : Two independent variables.

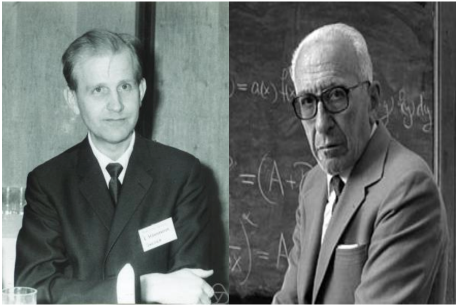
Local Carleman estimates

- Let u be a solution of

$$P(x, D)u = \sum_{|\alpha| \leqslant m} a_\alpha(x) \partial^\alpha u = 0$$

- Let $\psi \in C^2$ real valued function, $\nabla \psi \neq 0$.
- $(u \equiv 0 \quad in\, \{\psi < 0\}) \Rightarrow (u \equiv 0 \quad in\, \{\psi > 0\})$

Hörmander, Calderon,...



Local Carleman estimates

Let $P(x, D)$ be a linear differential operator of order 2 defined in an open set $\Omega \subset \mathbb{R}^n$. The following type of a priori estimate is called Carleman estimate

$$\tau \int_\Omega (|\nabla u|^2 + \tau^2 |u|^2) e^{2\tau\varphi} dx \leqslant C \int_\Omega |P(x, D)u|^2 e^{2\tau\varphi} dx,$$

for any $u \in \mathscr{C}_0^2(\Omega)$.

- Hörmander, Isakov, Lerner, Robbiano, Zuily...
- If the strong pseudo-convexity condition is satisfied, then we have the Carleman estimate.

Global Carleman estimates

Let $P(x, D)$ be a linear differential operator of order 2 defined in an open set $\Omega \subset \mathbb{R}^n$. The following type of a priori estimate is called global Carleman estimate

$$\tau \int_{\Omega} (|\nabla u|^2 + \tau^2 |u|^2) e^{2\tau\varphi} \leqslant \int_{\Omega} |P(x, D)u|^2 e^{2\tau\varphi} + \tau \int_{\Gamma_0} |\partial_\nu u|^2 e^{2\tau\varphi},$$

for any $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Here $\Gamma_0 \subset \Gamma = \partial\Omega$

Sufficient Conditions

We are interested in conditions on $\varphi = e^{\beta\psi}$ and $\underline{\underline{\Gamma_0 \subset \Gamma}}$ implying the Carleman estimate

$$\tau \int_\Omega (|\nabla u|^2 + \tau^2 |u|^2) e^{2\tau\varphi} \leqslant \int_\Omega |P(x,D)u|^2 e^{2\tau\varphi} + \tau \int_{\Gamma_0 \subseteq \Gamma} |\partial_\nu u|^2 e^{2\tau\varphi},$$

for any $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

- φ satisfies the strong pseudo-convexity condition in Ω
- φ satisfies the strong Lopatinskii condition on $\Gamma \backslash \Gamma_0$.
- Tataru, Imanuvilov, Isakov, Bellassoued,...

Example 1: Elliptic equation

Let $A \in L^\infty(\Omega, \mathbf{C}^n)$, $q \in L^\infty(\Omega, \mathbf{C})$, $\varphi = e^{\beta\psi}$

$$P(x, D) = -\Delta + A \cdot \nabla + q(x).$$

$$\underbrace{|\nabla\psi(x)| > 0 \quad \forall x \in \Omega,}_{\text{Pseudo-convexity in } \Omega} \quad \text{and} \quad \underbrace{\partial_\nu \psi(x) < 0, \quad x \in \Gamma\backslash\Gamma_0}_{\text{Lopatinskii in } \Gamma\backslash\Gamma_0} \quad (*)$$

Imanuvilov-Fursikov

Let $\Gamma_0 \subset \Gamma$ be an arbitrary open set. Then there exist $\psi \in \mathscr{C}^2(\overline{\Omega})$ s.t. $(*)$ is satisfied.

Example 1: Elliptic equation

- Let $A \in L^\infty(\Omega, \mathbb{C}^n)$, $q \in L^\infty(\Omega, \mathbb{C})$,

$$P(x,D)u = (-\Delta + A \cdot \nabla + q(x))u = f, \quad u \in H^2(\Omega) \cap H_0^1(\Omega),$$

- Let $\Gamma_0 \subset \Gamma$ be an arbitrary open set. there exist $C > 0$

$$C \|u\|_{H^1(\Omega)} \leqslant \|f\|_{L^2(\Omega)} + \|\partial_\nu u\|_{L^2(\Gamma_0)} .$$

- Let $\omega \subset \Omega$ be an arbitrary open set. there exist $C > 0$

$$C \|u\|_{H^1(\Omega)} \leqslant \|f\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)} .$$

Exemple 2: Wave equation

Let $\alpha > 0$, $Q = \Omega \times (0,T)$, $Q_\alpha = \Omega \times (\alpha, T-\alpha)$, $\Sigma = \Gamma \times (0,T)$,

$$P(x,D) = \partial_t^2 - \Delta + A \cdot \nabla + q(x), \qquad \text{(Wave)}$$

Determine $\Gamma_0 \subset \Gamma$ and $\Phi : \mathbb{R} \to \mathbb{R}$ such that

$$\|u\|_{H^1(Q_\alpha)} \leqslant \Phi\left(\|f\|_{L^2(Q)} + \|\partial_\nu u\|_{L^2(\Gamma_0 \times (0,T))} \right), \quad \Phi(0) = 0,$$

for any $u \in H^2(Q)$ such that $P(x,D)u = f$ and $u_{|\Sigma} = 0$.

Exemple 2: Wave Equation

Let $T > 0$, $Q = \Omega \times (0, T)$, $\Sigma = \Gamma \times (0, T)$

$$P(x, D) = \partial_t^2 - \Delta + A \cdot \nabla + q(x)$$

Let $\psi(x) = |x - x_0|^2$, $x_0 \in \mathbb{R}^n \backslash \overline{\Omega}$

$$\varphi(t, x) = e^{\beta(\psi(x) - \gamma t^2)}, \quad \Gamma_0 \supset \{x \in \Gamma, \ (x - x_0) \cdot \nu \geqslant 0\}$$

$T > 2\mathrm{Diam}(\Omega)$ we have

$$\|u\|_{H^1(Q_\alpha)} \leqslant \|u\|_{H^1(Q)}^{1-\mu} \left(\|f\|_{L^2(Q)} + \|\partial_\nu u\|_{L^2(\Gamma_0 \times (0,T))} \right)^\mu, \ \mu \in (0, 1)$$

for any $u \in H^2(Q)$ such that $P(x, D)u = f$ and $u_{|\Sigma} = 0$.

Wave equation: with variable coefficients

Let $(\mathcal{M}, \mathrm{g})$ a Riemannian manifold, $\Delta = \Delta_{\mathrm{g}}$. We assume that there exists a positive and smooth function ψ_0 on $\mathcal{M}$:

(A.1): ψ_0 is strictly convex on $\mathcal{M}$ with respect to the metric g:

$$D^2\psi_0(X,X)(x) > 0, \quad x \in \mathcal{M}, \quad X \in T_x\mathcal{M} \setminus \{0\}.$$

(A.2): We assume that $\psi_0(x)$ has no critical points on $\mathcal{M}$:

$$\min_{x\in\mathcal{M}} |\nabla\psi_0(x)| > 0.$$

(A.3): Under (A.1)-(A.2), let $\Gamma_0 \subset \partial\mathcal{M}$ satisfy

$$\{x \in \partial\mathcal{M};\ \partial_\nu\psi_0 \geqslant 0\} \subset \Gamma_0.$$

Wave equation: with variable coefficients

Let us define

$$\psi(t,x) = \psi_0(x) - \beta\,(t-t_0)^2 + \beta_0, \quad 0 < \beta < \varrho, \quad 0 < t_0 < T, \quad \beta_0 \geqslant 0,$$

We define the weight function $\varphi : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x,t) = e^{\gamma\psi(x,t)},$$

Theorem (Bellassoued 04')

Assume (A.1), (A.2) and (A.3) then the global Carleman estimates is hold in $(\mathcal{M}, \mathrm{g})$.

Wave equation: with variable coefficients

$$(A.1),(A.2),(A.3) \quad \begin{array}{c}\Rightarrow\\ \not\Leftarrow\end{array} \quad \text{GCC: Bardos-Lebeau-Rauch}$$

Wiffle Ball: $\mathbb{S}^2$ deleting ε-neighborhood of the segments along the equator with longitudes $[\pi/6,\pi/2]$, $[5\pi/6,7\pi/6]$ and $[3\pi/2,11\pi/6]$

Sufficient Conditions on Γ_0

Elliptic operator

We have a Lipschitz stability estimate of the Cauchy problem if $\Gamma_0 \subset \Gamma$ be an arbitrary open set.

Hyperbolic operator

We have a Hölder stability estimate of the Cauchy problem if $\Gamma_0 \subset \Gamma$ is large:

$$\Gamma_0 \supset \{x \in \partial\Omega;\ \partial_\nu \psi_0 \geqslant 0\}$$

Fourier-Bros-Iagolnitzer transform (F.B.I)

- We will specialize to the FBI transform with a Gaussian window: $v \in \mathscr{S}'(\mathbb{R})$,

$$\widetilde{\mathscr{F}}_\lambda v(s,t) = \alpha \int_{\mathbb{R}} e^{\frac{\lambda}{2}(2i(s-\eta)t-(s-\eta)^2)} v(\eta) d\eta.$$

- We also consider the closely related to Bargmann transform, defined by

$$\mathscr{F}_\lambda v(z) = \int_{\mathbb{R}} e^{-\frac{\lambda}{2}(z-\eta)^2} v(\eta) d\eta, \quad z = s - it$$

- $i\partial_s \left(\mathscr{F}_\lambda v(s,t)\right) = \mathscr{F}_\lambda(\partial_t v)(s,t)$

Fourier-Bros-Iagolnitzer transform (F.B.I)

$$-\underbrace{(\partial_s^2 + \Delta)}_{\text{Elliptic eq.}}(\mathscr{F}_\lambda u(s,t;x)) = \mathscr{F}_\lambda((\underbrace{\partial_t^2 - \Delta)u}_{\text{Wave eq.}})(s,t;x) := \mathscr{F}_\lambda(f)(t,s;x)$$

Let $\Gamma_0 \subset \Gamma$ be an arbitrary open set.

By the elliptic Lipschitz stability estimate:

$$\|\mathscr{F}_\lambda u(t,.)\|_{H^1(Q_\alpha)} \leqslant \Big(\|\mathscr{F}_\lambda f(t,.)\|_{L^2(Q)} + \|\partial_\nu \mathscr{F}_\lambda u(t,.)\|_{L^2(\Gamma_0\times(0,T))}\Big),$$

Fourier-Bros-Iagolnitzer transform (F.B.I)

$$\|\mathscr{F}_\lambda f(t,.)\|_{L^2(Q)} \leqslant e^{C\lambda} \|f\|_{L^2(Q)},$$
$$\|\partial_\nu \mathscr{F}_\lambda u(t,.)\|_{L^2(\Gamma_0\times(0,T))} \leqslant e^{C\lambda} \|\partial_\nu u\|_{L^2(\Gamma_0\times(0,T))}$$

$$\|\mathscr{F}_\lambda u(t,.)\|_{H^1(Q_\alpha)} \leqslant e^{C\lambda} \left(\|f\|_{L^2(Q)} + \|\partial_\nu u\|_{L^2(\Gamma_0\times(0,T))}\right)$$

Moreover

$$\|\mathscr{F}_\lambda u(t,.) - u\|_{H^1(Q_\alpha)} \leqslant \frac{C}{\lambda} \|u\|_{H^2(Q)}.$$

Then we have

$$\|u\|_{H^1(Q_\alpha)} \leqslant \frac{C}{\lambda} \|u\|_{H^2(Q)} + e^{C\lambda} \left(\|f\|_{L^2(Q)} + \|\partial_\nu u\|_{L^2(\Gamma_0\times(0,T))}\right)$$

Fourier-Bros-Iagolnitzer transform (F.B.I)

Let $\Gamma_0 \subset \Gamma$ be an arbitrary open set, we have For the Wave equation:

$$\|u\|_{H^1(Q_\alpha)} \leqslant \Phi\left(\|f\|_{L^2(Q)} + \|\partial_\nu u\|_{L^2(\Gamma_0\times(0,T))}\right),$$

$$\Phi(\delta) = C(\log(2+\delta^{-1}))^{-1}.$$

Outline

Setting at a boundary

We consider

- P be a smooth elliptic of order $m = 2\mu$.

$$P = \sum_{|\alpha| \leqslant m} a_\alpha(x)\partial^\alpha,$$

  with complex-valued coefficients.

- $m/2$ linear smooth boundary operators of order less than m

$$B^k = \sum_{|\alpha| \leqslant \beta_k} b^k_\alpha(x)\partial^\alpha, \quad k = 1, \ldots, \mu = m/2,$$

  with complex-valued coefficients, defined in some neighborhood of $\partial\Omega$.

Consider the elliptic boundary value problem

$$\begin{cases} Pu(x) = f(x), & x \in \Omega, \\ B^k u(x) = g^k(x), & x \in \partial\Omega, \quad k = 1, \ldots, \mu. \end{cases}$$

Setting at a boundary

Consider the elliptic boundary value problem

$$\begin{cases} Pu(x) = f(x), & x \in \Omega, \\ B^k u(x) = g^k(x), & x \in \partial\Omega, \quad k = 1, \dots, \mu. \end{cases}$$

We wish to obtain an estimate of the form

$$\|e^{\tau\varphi}u\|^2 + |e^{\tau\varphi}T(u)|^2 \lesssim \|e^{\tau\varphi}P(x,D)u\|^2 + \sum_{k=1}^{\mu} |e^{\tau\varphi}B^k(x,D)u|^2,$$

for u supported near a point at the boundary $T(u)$ is the trace of $(u, D_\nu u, \dots, D_\nu^{m-1}u)$. If we set

$$P_\varphi = e^{\tau\varphi}P(x,D)e^{-\tau\varphi}; \quad B^k_\varphi = e^{\tau\varphi}B^k(x,D)e^{-\tau\varphi}; \quad v = e^{\tau\varphi}u$$

then the Carleman estimate reads:

$$\|v\|^2 + |T(v)|^2 \lesssim \|P_\varphi v\|^2 + \sum_{k=1}^{\mu} |B^k_\varphi v|^2,$$

Estimates of this form were obtained by Tataru. We give more precise estimates here and include the complex coefficient case.

1st order operator at a boundary

Let $\rho^+ \in \mathbf{C}$ such that $\underline{\mathrm{Im}\rho^+ > 0}$, $f \in \mathscr{S}(\mathbf{R}^+)$ and $u_0 \in \mathbf{C}$. We consider the boundary problem

$$\begin{cases} (D_{x_n} - \rho^+)\, u(x_n) = f(x_n) \quad x_n > 0 & (1^+) \\ u(0) = u_0 \in \mathbf{C} & (2) \end{cases}$$

The first line can be solved by Fourier transformation: Let f^+ the extension of f by 0 on $(-\infty, 0)$. Let

$$v(x_n) = \mathscr{F}^{-1}\left(\frac{1}{\xi_n - \rho^+}\mathscr{F}(f^+)\right)$$

then $\underline{v}(x_n)$ the restriction of v to $\mathbf{R}^+$ is a solution of the first line. Since $\mathrm{Im}\rho^+ > 0$ we have

$$\begin{aligned} \underline{v}(x_n) &= \mathscr{F}^{-1}\left(\frac{1}{\xi_n - \rho^+}\right) * f^+(x_n) \\ &= i\left(H(x_n)e^{i\rho^+ x_n}\right) * f^+(x_n) = i\int_0^{x_n} e^{i\rho^+(x_n - y_n)} f(y_n)\,dy_n \end{aligned}$$

1st order operator at a boundary

Let $\rho^+ \in \mathbf{C}$ such that $\underline{\mathrm{Im}\rho^+ > 0}$, $f \in \mathscr{S}(\mathbf{R}^+)$ and $u_0 \in \mathbf{C}$. We consider the boundary problem

$$\begin{cases} (D_{x_n} - \rho^+)\, u(x_n) = f(x_n) \quad x_n > 0 & (1^+) \\ u(0) = u_0 \in \mathbf{C} & (2) \end{cases}$$

The full problem is then solved by reduction to a semihomogenous problem: Set $w(x_n) = u(x_n) - v(x_n)$ then since $v(0) = 0$, $w(x_n)$ must solve

$$\begin{cases} (D_{x_n} - \rho^+)\, w(x_n) = 0 \quad x_n > 0 \\ w(0) = u_0 \in \mathbf{C} \end{cases}$$

that is

$$w(x_n) = e^{i\rho^+ x_n} u_0$$

So the solution of the full problem is

$$u(x_n) = \underbrace{e^{i\rho^+ x_n} u_0}_{\in \mathscr{S}(\mathbf{R}^+)} + \underbrace{i \int_0^{x_n} e^{i\rho^+(x_n - y_n)} f(y_n)\, dy_n}_{\in \mathscr{S}(\mathbf{R}^+)}$$

1st order operator at a boundary

Let $\rho^+ \in \mathbf{C}$ such that $\underline{\mathrm{Im}\rho^+ > 0}$, $f \in \mathscr{S}(\mathbf{R}^+)$ and $u_0 \in \mathbf{C}$. We consider the boundary problem

$$\left\{ \begin{array}{ll} (D_{x_n} - \rho^+)\, u(x_n) = f(x_n) \quad x_n > 0 & \qquad (1^+) \\ u(0) = u_0 \in \mathbf{C} & \qquad (2) \end{array} \right.$$

The full problem is then solved by reduction to a semihomogenous problem: Set $w(x_n) = u(x_n) - v(x_n)$ then since $v(0) = 0$, $w(x_n)$ must solve

$$\left\{ \begin{array}{l} (D_{x_n} - \rho^+)\, w(x_n) = 0 \quad x_n > 0 \\ w(0) = u_0 \in \mathbf{C} \end{array} \right.$$

that is

$$w(x_n) = e^{i\rho^+ x_n} u_0$$

So the solution of the full problem is

$$u(x_n) = \underbrace{e^{i\rho^+ x_n} u_0}_{\in \mathscr{S}(\mathbf{R}^+)} + \underbrace{i \int_0^{x_n} e^{i\rho^+ (x_n - y_n)} f(y_n) dy_n}_{\in \mathscr{S}(\mathbf{R}^+)}$$

1st order operator at a boundary

Let $\rho^- \in \mathbf{C}$ such that $\underline{\mathrm{Im}\rho^- < 0}$, $f \in \mathscr{S}(\mathbf{R}^+)$ and $u_0 \in \mathbf{C}$. We consider the boundary problem

$$\left(D_{x_n} - \rho^-\right) u(x_n) = f(x_n) \quad x_n > 0 \qquad (1^-)$$

One solution of (1^-) is found by taking the restriction to $(0, \infty)$ of the L^2 function:

$$\begin{aligned} \underline{v}(x_n) &= \mathscr{F}^{-1}\left(\frac{1}{\xi_n - \rho^-}\right) * f^+(x_n) \\ &= \left(-iH(-x_n)e^{i\rho^- x_n}\right) * f^+(x_n) = -i\int_{x_n}^{\infty} e^{i\rho^-(x_n - y_n)} f(y_n)\,dy_n \end{aligned}$$

it is locally absolutely continous. Now $\underline{v}(x_n)$ is the only solution belonging to $L^2(\mathbf{R}^+)$: for any other solution $u(x_n)$ let $w(x_n) = u(x_n) - v(x_n)$ then:

$$\left(D_{x_n} - \rho^-\right) w(x_n) = 0 \quad x_n > 0$$

which is of the form $w(x_n) = e^{i\rho^- x_n} c_0$, that is not in $L^2(\mathbf{R}^+)$ for $c_0 \neq 0$ since $\mathrm{Im}\rho^- < 0$.

1st order operator at a boundary

We give special names to the occuring operators:

$$\lambda^+(\xi_n) = \frac{1}{\xi_n - \rho^+} \quad \text{a symbol in } S^{-1}(\mathbf{R}\times\mathbf{R})$$

$$\Lambda^+(D_n)f = \mathscr{F}^{-1}\left(\lambda^+(\xi_n)\mathscr{F}(f)\right) \quad \text{the corresponding } \Psi d.o$$

$$\Lambda^+_\Omega(D_n)f = r^+\mathscr{F}^{-1}\left(\lambda^+(\xi_n)\mathscr{F}(f^+)\right) \quad \text{its restriction to } \Omega = \mathbf{R}^+$$

Let us also introduce the multiplication operotor

$$k(x_n,\rho^+)c = H(x_n)e^{i\rho^+x_n}c, \quad c\in\mathbf{C}.$$

Then the conclusions that the operator:

$$\begin{array}{rcccc} A_+ & : & \mathscr{S}(\mathbf{R}^+) & \to & \mathscr{S}(\mathbf{R}^+)\times\mathbf{C} \\ & & u & \mapsto & (f=(D_n-\rho^+)u, u(0)) \end{array}$$

is bijective and has the inverse operator

$$\begin{array}{rcccc} A_+^{-1} & : & \mathscr{S}(\mathbf{R}^+)\times\mathbf{C} & \to & \mathscr{S}(\mathbf{R}^+) \\ & & (f,u_0) & \mapsto & k^+(x_n)u_0 + \Lambda^+_\Omega(D_n)f \end{array}$$

1st order operator at a boundary

$$\lambda^-(\xi_n) = \frac{1}{\xi_n - \rho^-} \quad \text{a symbol in } S^{-1}(\mathbf{R} \times \mathbf{R})$$

$$\Lambda^-(D_n)f = \mathscr{F}^{-1}\left(\lambda^-(\xi_n)\mathscr{F}(f)\right) \quad \text{the corresponding } \Psi d.o$$

$$\Lambda^-_\Omega(D_n)f = r^+\mathscr{F}^{-1}\left(\lambda^-(\xi_n)\mathscr{F}(f^+)\right) \quad \text{its restriction to } \Omega = \mathbf{R}^+$$

Then the conclusions from the example with ρ^- is that the operator:

$$\begin{array}{rcccc} A_- & : & \mathscr{S}(\mathbf{R}^+) & \to & \mathscr{S}(\mathbf{R}^+) \\ & & u & \mapsto & f = (D_n - \rho^-)u \end{array}$$

is bijective and has the inverse operator

$$\begin{array}{rcccc} A_-^{-1} & : & \mathscr{S}(\mathbf{R}^+) & \to & \mathscr{S}(\mathbf{R}^+) \\ & & f & \mapsto & \Lambda^-_\Omega(D_n)f \end{array}$$

2nd order operator at a boundary

The general theory for elliptic boundary value problems can be regarded as built up from the laste two examples. Consider the Dirichlet problem on the half space $\mathbf{R}^n_+$

$$\begin{cases} (-\Delta + 1)\, u(x) = f(x) & x_n > 0 \\ u(x', 0) = u_0(x') & x_n = 0 \end{cases}$$

By Fourier transform in the x'-varibale

$$(1) \qquad \begin{cases} \left(D_{x_n}^2 + \langle \xi' \rangle^2\right) \hat{u}(\xi', x_n) = \hat{f}(\xi', x_n)) & x_n > 0 \\ \hat{u}(\xi', 0) = \hat{u}_0(\xi') & x_n = 0 \end{cases}$$

Now

$$\left(D_{x_n}^2 + \left\langle \xi' \right\rangle^2\right) = \left(D_{x_n} - i \left\langle \xi' \right\rangle\right)\left(D_{x_n} + i \left\langle \xi' \right\rangle\right) = \left(D_{x_n} - \rho^+\right)\left(D_{x_n} - \rho^-\right)$$

We can solve (1) by breaking it up in two problems

$$\begin{cases} (D_{x_n} - \rho^+)\, \hat{u}(x_n) = v & x_n > 0 \\ \hat{u}(0) = \hat{u}_0(\xi') & x_n = 0 \end{cases} ; \quad \left(D_{x_n} - \rho^-\right) v = \hat{f} \quad x_n > 0$$

2nd order operator at a boundary

We can solve (1) by breaking it up in two problems

$$\begin{cases} \left(D_{x_n} - \rho^{+}\right) \hat{u}(x_n) = v & x_n > 0 \\ \hat{u}(0) = \hat{u}_0(\xi') & x_n = 0 \end{cases} \quad ; \quad \left(D_{x_n} - \rho^{-}\right) v = \hat{f} \quad x_n > 0$$

$$v = A_{-}^{-1}\hat{f}(\xi', x_n) = \Lambda_{\Omega}^{-}(D_n)\hat{f}(\xi', x_n), \quad \rho^{-} = -i\left\langle \xi' \right\rangle$$

$$\hat{u}(\xi', x_n) = A_{+}^{-1}(v(\xi', \cdot), \hat{u}_0(\xi')) = k^{+}(x_n)\hat{u}_0(\xi')) + \Lambda_{\Omega}^{+}(D_n)v, \quad \rho^{+} = i\left\langle \xi' \right\rangle$$

$$\begin{aligned} \hat{u}(\xi', x_n) &= A_{+}^{-1}\left(v(\xi', \cdot), \hat{u}_0(\xi')\right) \\ &= k^{+}(x_n)\hat{u}_0(\xi') + \Lambda_{\Omega}^{+}(D_n)\Lambda_{\Omega}^{-}(D_n)\hat{f}(\xi', x_n) \end{aligned}$$

2nd order operator at a boundary

We can solve (1) by breaking it up in two problems

$$\begin{cases} (D_{x_n} - \rho^+)\,\hat{u}(x_n) = v & x_n > 0 \\ \hat{u}(0) = \hat{u}_0(\xi') & x_n = 0 \end{cases} \;; \quad \left(D_{x_n} - \rho^-\right) v = \hat{f} \quad x_n > 0$$

$$v = A_-^{-1}\hat{f}(\xi', x_n) = \Lambda_\Omega^-(D_n)\hat{f}(\xi', x_n), \quad \rho^- = -i\left\langle \xi' \right\rangle$$

$$\hat{u}(\xi', x_n) = A_+^{-1}(v(\xi', \cdot), \hat{u}_0(\xi')) = k^+(x_n)\hat{u}_0(\xi')) + \Lambda_\Omega^+(D_n)v, \quad \rho^+ = i\left\langle \xi' \right\rangle$$

$$\begin{aligned} \hat{u}(\xi', x_n) &= A_+^{-1}\left(v(\xi', \cdot), \hat{u}_0(\xi')\right) \\ &= k^+(x_n)\hat{u}_0(\xi') + \Lambda_\Omega^+(D_n)\Lambda_\Omega^-(D_n)\hat{f}(\xi', x_n) \end{aligned}$$

2nd order operator at a boundary

We can solve (1) by breaking it up in two problems

$$\begin{cases} (D_{x_n} - \rho^+)\,\hat{u}(x_n) = v & x_n > 0 \\ \hat{u}(0) = \hat{u}_0(\xi') & x_n = 0 \end{cases} \quad ; \quad \left(D_{x_n} - \rho^-\right) v = \hat{f} \quad x_n > 0$$

$$v = A_-^{-1}\hat{f}(\xi', x_n) = \Lambda_\Omega^-(D_n)\hat{f}(\xi', x_n), \quad \rho^- = -i\left\langle \xi' \right\rangle$$

$$\hat{u}(\xi', x_n) = A_+^{-1}(v(\xi', \cdot), \hat{u}_0(\xi')) = k^+(x_n)\hat{u}_0(\xi')) + \Lambda_\Omega^+(D_n)v, \quad \rho^+ = i\left\langle \xi' \right\rangle$$

$$\begin{aligned} \hat{u}(\xi', x_n) &= A_+^{-1}\left(v(\xi', \cdot), \hat{u}_0(\xi')\right) \\ &= k^+(x_n)\hat{u}_0(\xi') + \Lambda_\Omega^+(D_n)\Lambda_\Omega^-(D_n)\hat{f}(\xi', x_n) \end{aligned}$$

2nd order operator at a boundary

2nd-order operators at the boundary were precisely treated by G. Lebeau and L. Robbiano (95, 97).

Set

$$P=-\Delta=\sum_j D_j^2; \quad \partial\Omega=\{x_n=0\}, \quad \Omega=\{x_n>0\}$$

We write $x=(x',x_n)$ and $\xi=(\xi',\xi_n)$.

Take $\varphi=\varphi(x_n)$ such that $\varphi'>0$. Then

$$\begin{aligned} P_\varphi &= \big(D_{x_n}+i\tau\varphi'(x)\big)^2+\overbrace{\sum_{1\leqslant j\leqslant n-1} D_j^2}^{D'^2} \\ &= \big(D_{x_n}+i(\tau\varphi'(x)+\big|D'\big|)\big)\big(D_{x_n}+i(\tau\varphi'(x)-\big|D'\big|)\big), \end{aligned}$$

where $|D'|=\mathrm{Op}(|\xi'|)$.

Consider the principal symbol

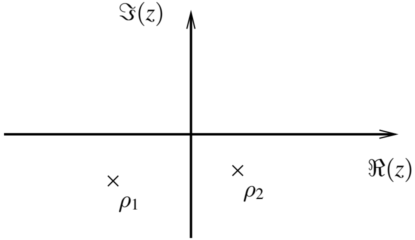
$$p_\varphi(x,\xi)=p(x,\xi+i\tau\varphi')=\big(\xi_n+i(\tau\varphi'(x)+|\xi'|)\big)\big(\xi_n+i(\tau\varphi'(x)-|\xi'|)\big)$$

2nd order operator at a boundary

Principal symbol

$$p_\varphi(x,\xi) = p(x,\xi + i\tau\varphi') = \big(\xi_n + i(\tau\varphi'(x) + |\xi'|)\big)\big(\xi_n + i(\tau\varphi'(x) - |\xi|)\big)$$
$$= \big(\xi_n - \rho_1\big)\big(\xi_n - \rho_2\big)$$

In the low-frequency regime, $|\xi'|$ small,



We have (a microlocal perfect elliptic estimate)

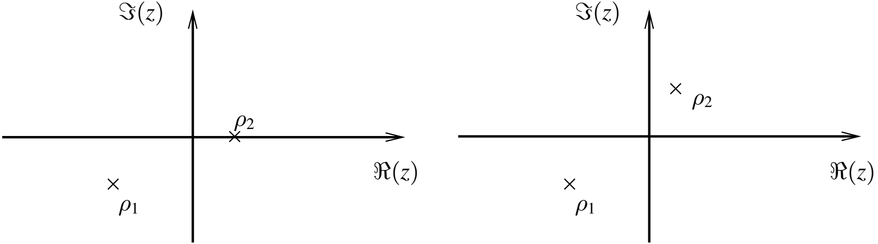
$$\|v\|_{2,\tau} + |T(v)|_{1,1/2,\tau} \lesssim \|P_\varphi v\|_{L^2} \quad (+\cdots)$$

2nd order operator at a boundary

Principal symbol

$$p_\varphi(x,\xi) = p(x,\xi + i\tau\varphi') = \big(\xi_n + i(\tau\varphi'(x) + |\xi'|)\big)\big(\xi_n + i(\tau\varphi'(x) - |\xi|)\big)$$
$$= \big(\xi_n - \rho_1\big)\big(\xi_n - \rho_2\big)$$

In the high-frequency regime, $|\xi'|$ large,



We have

$$\tau^{-1/2}\,\|v\|_{2,\tau} + |T(v)|_{1,1/2,\tau} \lesssim \|P_\varphi v\|_{L^2} + \text{ boundary norm} \quad (+\cdots)$$

The boundary norm can be of Dirichlet, Neumann, Robin type...

High-order operator at a boundary

Set $\varrho' = (x, \xi', \tau)$
We have

$$p_\varphi(\varrho', \xi_n) = \prod_{j=1}^{m} \big(\xi_n - \rho_j(\varrho')\big) = p_\varphi^+(\varrho', \xi_n) p_\varphi^-(\varrho', \xi_n) p_\varphi^0(\varrho', \xi_n),$$

with

$$p_\varphi^{\pm}(\varrho', \xi_n) = \prod_{\pm \mathrm{Im}\rho_j > 0} (\xi_n - \rho_j), \quad p_\varphi^{0}(\varrho', \xi_n) = \prod_{\mathrm{Im}\rho_j = 0} (\xi_n - \rho_j).$$

p_φ^- yields a prefect elliptic estimate.

We set

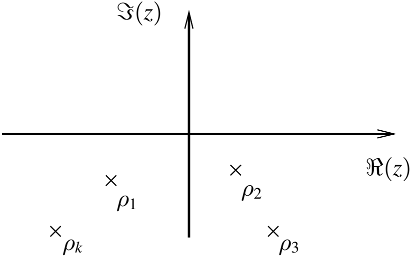
$$\kappa_\varphi(\varrho', \xi_n) = p_\varphi^+(\varrho', \xi_n) p_\varphi^0(\varrho', \xi_n)$$

High-order operator at a boundary: $\kappa_\varphi(\varrho', \xi_n) = 1$

Principal symbol

$$p_\varphi(x,\xi) = p(x, \xi + i\tau\varphi') = \prod_{j=1}^{k} \left(\xi_n - \rho_j(x,\tau,\xi')\right)$$

If all the roots have a negative imaginary part,



Write $p_\varphi = a + ib$, a and b both self adjoint and $A = a(x, D, \tau)$, $B = b(x, D, \tau)$, we have

$$\|P_\varphi(x,D)v\|_{L^2}^2 = \|Av\|_{L^2}^2 + \|Bv\|_{L^2}^2 + 2\Re\left(Av, iBv\right)$$

Bézout Matrices

Given two univariate polynomials $a(\zeta) = \sum_{j=0}^{m} a_j \zeta^j$, $b(\zeta) = \sum_{j=0}^{m} b_j \zeta^j$, we build the following bivariate polynomial

$$B_{a,b}(\zeta, \widetilde{\zeta}) = \frac{a(\zeta)b(\widetilde{\zeta}) - a(\widetilde{\zeta})b(\zeta)}{\zeta - \widetilde{\zeta}} = \sum_{j,k=0}^{m-1} g_{j,k}\, \zeta^j \widetilde{\zeta}^k,$$

called the Bézoutian of a and b, and the corresponding symmetric matrix $g_{a,b} = (g_{j,k})$ of size $m \times m$ with entries $g_{j,k}$, bilinear in the coefficients of a and b, is called the Bézout matrix and given by :

$$g_{j,k} = \sum_{\ell=0}^{\min(j,k)} \left(b_\ell a_{j+k-\ell+1} - b_{j+k-\ell+1} a_\ell \right),$$

upon letting $a_k = b_k := 0$ for $k > m$ and $k < 0$. With this Bézout matrix we associate the following bilinear form:

$$\widetilde{B}_{a,b}(\mathbf{z}, \mathbf{z}') = \sum_{j,k=0}^{m-1} g_{j,k} z_j \overline{z}'_k, \quad \mathbf{z} = (z_0, \dots, z_{m-1}),\ \mathbf{z}' = (z'_0, \dots, z'_{m-1}) \in \mathbf{C}^m.$$

Hermite Theorem

The following Hermite Theorem providing a relation between the roots of a polynomial and the Bézout matrix associated with the real and imaginary parts of the polynomial.

Lemma (Hermite Theorem)

Let $h(\zeta) = a(\zeta) + ib(\zeta)$ *be a polynomial of degree* $k \geqslant 1$, *where* $a(\zeta)$ *and* $b(\zeta)$ *are polynomials with real coefficients. Assume that all the roots of* $h(\zeta)$ *are in the lower complex half-plane* $\{\Im\zeta < 0\}$. *Then the roots of* $a(\zeta)$ *and* $b(\zeta)$ *are real and distinct. Moreover, the bilinear form* $\widetilde{B}_{a,b}(\mathbf{z}, \mathbf{z}')$ *is positive, that is there exists* $C > 0$ *such that*

$$\widetilde{B}_{a,b}(\mathbf{z}, \mathbf{z}) \geqslant C\,|\mathbf{z}|^2, \quad \mathbf{z} \in \mathbf{C}^k.$$

A generalized Green formula

Consider two smooth and real symbols $a(x, \xi, \tau)$ and $b(x, \xi, \tau)$. The following identity holds true

$$2\mathrm{Re}\,(Av, iBv) = H_{a,b}(v) + \mathscr{B}_{a,b}(v) + R(v), \quad A = a(x, D, \tau),\ B = b(x, D, \tau),$$

for any $v \in \mathscr{S}(\mathbf{R}^n_+)$. Here:

- $\mathscr{B}_{a,b}$ is the boundary quadratic form with symbol the Bézout matrix $\widetilde{B}_{a,b}(\mathbf{z}, \mathbf{z}')$.
- $H_{a,b}$ is an interior quadratic form with real symbol

$$h_{a,b}(\varrho) = \mathrm{sub}(a, b)(\varrho) = \{a, b\} + \sum_{|\alpha|=1} \left(b\partial_\xi^\alpha \partial_x^\alpha a - a\partial_\xi^\alpha \partial_x^\alpha b\right).$$

- the remainder term $R(v)$ is a quadratic form that satisfies

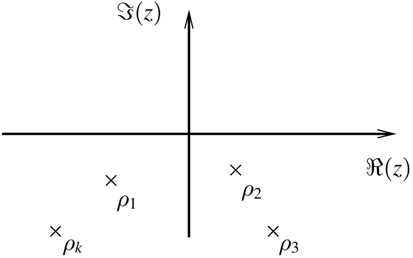
$$|R(v)| \leqslant C \|v\|^2_{k,-1,\tau}\,.$$

High-order operator at a boundary: $\kappa_\varphi(\xi_n) = 1$

Principal symbol

$$p_\varphi(x,\xi) = p(x,\xi + i\tau\varphi') = \prod_{j=1}^{k} \big(\xi_n - \rho_j(x,\tau,\xi')\big)$$

If all the roots have a negative imaginary part,



we want to prove (a microlocal perfect elliptic estimate)

$$\|v\|_{m,\tau} + |T(v)|_{k-1,1/2,\tau} \lesssim \|p_\varphi v\|_{L^2} \quad (+\cdots)$$

Ideas of the proof

We have

$$2\Re\,(Av, iBv) = H_{a,b}(v) + \mathscr{B}_{a,b}(v) + R(v), \quad A = a(x, D, \tau),\ B = b(x, D, \tau),$$

Write $p_\varphi = a + ib$, a and b both self adjoint.

$$\|A(x, D, \tau)v\|_{L^2}^2 + \|B(x, D, \tau)v\|_{L^2}^2 \geqslant C\,\|v\|_{k,\tau}^2 - C'|T(v)|_{k-1,1/2,\tau}^2$$

(the roots of a and b are real and distinct)

With a generalized green formula we have

$$2\Re(Av, iBv)_{L^2} \geqslant H_{a,b}(v) + \mathscr{B}_{a,b}(v) - C\,\|v\|_{k,-1/2,\tau}^2$$

Sub-ellipticity property:

$$p_\varphi(\varrho, \xi_n) = 0 \quad \Rightarrow \quad \{a, b\}(\varrho, \xi_n) > 0 \quad \Rightarrow \quad h_{a,b}(\varrho, \xi_n) > 0.$$

The position of the roots and the Hermite Theorem give

$$\mathscr{B}_{a,b}(v) \gtrsim |T(v)|_{k-1,1/2,\tau}^2$$

High-order operator at a boundary

Set $\varrho' = (x, \xi', \tau)$
We have

$$p_\varphi(\varrho', \xi_n) = \prod_{j=1}^{m} \big(\xi_n - \rho_j(\varrho')\big) = p_\varphi^+(\varrho', \xi_n) p_\varphi^-(\varrho', \xi_n) p_\varphi^0(\varrho', \xi_n),$$

with

$$p_\varphi^{\pm}(\varrho', \xi_n) = \prod_{\pm \mathrm{Im}\rho_j > 0} (\xi_n - \rho_j), \quad p_\varphi^{0}(\varrho', \xi_n) = \prod_{\mathrm{Im}\rho_j = 0} (\xi_n - \rho_j).$$

p_φ^- yields a prefect elliptic estimate.

We set

$$\kappa_\varphi(\varrho', \xi_n) = p_\varphi^+(\varrho', \xi_n) p_\varphi^0(\varrho', \xi_n)$$

High-order operator at a boundary

Boundary operators: $B^k, \quad k = 1, \ldots, \mu$

Conjugated operators: $B^k_\varphi = e^{\tau\varphi} B^k e^{-\tau\varphi}$

Principal symbol: $b^k_\varphi(\varrho', \xi_n) \equiv b^k_\varphi(\xi_n)$

Strong Lopatinskii condition:
The set $\{b^k_\varphi(\xi_n)\}_{k=1,\ldots,\mu}$ is complete modulo $\kappa_\varphi(\xi_n)$ as polynomials in ξ_n.

For all $f(\xi_n)$ polynomial, there exist $c_1, \ldots, c_\mu \in \mathbf{C}$ and $q(\xi_n)$ polynomial such that

$$f(\xi_n) = \sum_{k=1}^{\mu} c_k b^k_\varphi(\xi_n) + q(\xi_n)\kappa_\varphi(\xi_n)$$

We have thus obtained

Theorem (Bellassoued, Le Rousseau)

Under

- *sub-ellipticity condition,*
- *strong Lopatinskii condition,*

Let $x_0 \in \partial\Omega$. *There exist* W *a nbhd of* x_0, $C > 0$, *and* $\tau_0 > 0$ *such that at the boundary*

$$\tau^{-1} \|e^{\tau\varphi} u\|^2_{m,\tau} + |e^{\tau\varphi} T(u)|^2_{m-1,1/2,\tau} \leqslant C\Big(\|e^{\tau\varphi} P(x,D)u\|^2_{L^2} + \sum_{k=1}^{\mu} |e^{\tau\varphi} B^k(x,D)u|^2_{m-1/2-\beta_k,\tau}\Big),$$

for $\tau \geqslant \tau_0$ *and* $u = w_{|\Omega}$ *with* $w \in \mathscr{C}^\infty_0(W)$.

Outline

Dynamic Schrödinger equation in waveguide

Let ω is an open connected bounded domain in $\mathbf{R}^{n-1}$, $n \geqslant 3$, with boundary $\partial\omega$, and we consider $\Omega := \omega \times \mathbf{R}$, in $\mathbf{R}^n$, with cross section ω. Its boundary is denoted by $\Gamma := \partial\omega \times \mathbf{R}$. Given $T > 0$, $p : \Omega \to \mathbf{R}$ and $u_0 : \Omega \to \mathbf{R}$, we consider the Schrödinger equation,

$$-i\partial_t u(x,t) - \Delta u(x,t) + p(x)u(x,t) = 0,\ (x,t) \in \Omega \times (0,T),$$

associated with the initial data u_0,

$$u(x,0) = u_0(x),\ x \in \Omega,$$

and the homogeneous Dirichlet boundary condition,

$$u(x,t) = 0,\ (x,t) \in \Gamma \times (0,T).$$

Dynamic Schrödinger equation in waveguide

Given an arbitrary relatively open subset $S_* \subset \partial\omega$, we aim for determining the unknown potential $p = p(x)$ from one Neumann observation of the function u_p on $\Sigma_* := \Gamma_* \times (0, T)$, where $\Gamma_* := S_* \times \mathbf{R}$ is an infinitely extended strip.

The uniqueness issue: is to know whether any two admissible potentials $p_j, j = 1, 2$, are equal, i.e. $p_1(x) = p_2(x)$ for a.e. $x \in \Omega$, if their observation data coincide, that is, if we have

$$\partial_\nu u_{p_1}(x,t) = \partial_\nu u_{p_2}(x,t),\ \ (x,t) \in \Sigma_*.$$

Dynamic Schrödinger equation in waveguide

Theorem (Bellassoued-Kian-Soccorsi)

Assume that

$$\exists\kappa > 0,\ \exists d_0,\ |u_0(x', x_n)| \geqslant \kappa\langle x_n\rangle^{-d_0/2},\ (x', x_n) \in \Omega.$$

For $p_j \in \mathcal{P}_{admissible}(p_0, \omega_0)$, $j = 1, 2$, we denote by u_j the solution to the IBVP, where p_j is substituted for p. Then, for any $\epsilon \in (0, 1)$, there exists a constant $C > 0$, such that we have

$$\|p_1 - p_2\|_{L^2(\Omega)} \leqslant C\left(\|\partial_\nu(u_1 - u_2)\|_* + |\log\|\partial_\nu(u_1 - u_2)\|_*|^{-1}\right)^\epsilon.$$

$$\|\partial_\nu u\|_* := \|\partial_\nu u\|_{H^1(0,T;L^2(\Gamma_*))},\ u \in \mathcal{H}^2.$$

Weak observability for the Schrödinger eq.

Let the linear Schrödinger equation

$$\begin{cases} i\partial_t v + \Delta v = 0 & \text{in} \quad \Omega \times [0, \infty), \\ v(x,t) = 0 & \text{on} \quad \partial\Omega \times [0, \infty), \\ v(x,0) = v_0(x) & \text{in} \quad \Omega \end{cases} \tag{1}$$

Theorem

Let Γ_0 be a non-empty open subset of $\partial\Omega$. For any $\mu \in (0,1)$ and $T > 0$, there exists $C > 0$ and $\lambda_0 > 0$ such that for any non-identically zero initial data $u_0 \in H^2(\Omega)$, we have

$$\|v_0\|^2_{L^2(\Omega)} \leqslant C\left[\frac{1}{\gamma^{2\mu}}\|v_0\|^2_{H^2(\Omega)} + e^{C\gamma}\int_0^T\int_{\Gamma_0} |\partial_\nu v(x,t)|^2\, dxdt\right]$$

for any $\gamma \geqslant \gamma_0$.

Weak observability for the Schrödinger eq.

Let the linear Schrödinger equation

$$\begin{cases} i\partial_t v + \Delta v = 0 & \text{in} \quad \Omega \times [0,\infty), \\ v(x,t) = 0 & \text{on} \quad \partial\Omega \times [0,\infty), \\ v(x,0) = v_0(x) & \text{in} \quad \Omega \end{cases} \tag{1}$$

Theorem

Let Γ_0 be a non-empty open subset of $\partial\Omega$. For any $\mu \in (0,1)$ and $T > 0$, there exists $C > 0$ and $\lambda_0 > 0$ such that for any non-identically zero initial data $u_0 \in H^2(\Omega)$, we have

$$\|v_0\|^2_{L^2(\Omega)} \leqslant C \left[\frac{1}{\gamma^{2\mu}} \|v_0\|^2_{H^2(\Omega)} + e^{C\gamma} \int_0^T \!\! \int_{\Gamma_0} |\partial_\nu v(x,t)|^2 \, dxdt \right]$$

for any $\gamma \geqslant \gamma_0$.

Short bibliography

1991 Robbiano: FBI Transform: Quantitave uniqueness theorem for the wave equation.

1995 Lebeau-Robbiano: Decay rate for the damped wave equation without GCC

1998 Tataru: UCP for PDEs with partial analytic coefficients

1998 Robbiano-Zuily: UCP PDEs with partial analytic coefficients

1999 Phung: Schrödinger equation

2004 Bellassoued, Application to inverse problem: wave equation

2006 Bellassoued-Yamamoto: Inverse problem for acoustic coefficient

2008 Bellassoued- Imanuvilov- Yamamoto: Stability estimates for the Lamé system

2009 Bellassoued, Choulli: Stability for the magnetic Shrödinger equation in the euclidean case.

2009 Bellassoued, Choulli Yamamoto: Stability estimates for the wave equation and Schrödinger equation Dirichlet-to-Neumann

Weak observability for the Schrödinger eq.

Step 1: Observability of the Heat eq.

Let $h > 0$, we consider the linear heat equation

$$\begin{cases} \partial_s w + \Delta w = f & \text{in} \quad \Omega \times (0,h), \\ w(x,s) = 0 & \text{on} \quad \partial\Omega \times (0,h), \end{cases} \tag{2}$$

Theorem

Let $\Gamma_0 \subset \partial\Omega$. Then there exists $C_h > 0$ such that

$$\int_\Omega |w(x,0)|^2 \, dx \leqslant C_h \left(\int_0^h \int_{\Gamma_0} |w(x,s)|^2 \, dxds + \int_0^h \int_\Omega |f(x,s)|^2 \, dxds \right)$$

Carleman parabolic estimates.

Weak observability for the Schrödinger eq.

Step 1: Observability of the Heat eq.

Let $h > 0$, we consider the linear heat equation

$$\begin{cases} \partial_s w + \Delta w = f & \text{in} \quad \Omega \times (0,h), \\ w(x,s) = 0 & \text{on} \quad \partial\Omega \times (0,h), \end{cases} \tag{2}$$

Theorem

Let $\Gamma_0 \subset \partial\Omega$. Then there exists $C_h > 0$ such that

$$\int_\Omega |w(x,0)|^2 \, dx \leqslant C_h \left(\int_0^h \int_{\Gamma_0} |w(x,s)|^2 \, dxds + \int_0^h \int_\Omega |f(x,s)|^2 \, dxds \right)$$

Carleman parabolic estimates.

Weak observability for the Schrödinger eq.

Step 2: Connection between the Schrödinger's and the heat eqs.
Let $\mu \in (0,1)$ and choose $m \in \mathbb{N}^*$ such that $0 < \mu + \frac{1}{2m} < 1$. Put $\rho = 1 - \frac{1}{2m} > \mu$. For any $\gamma \geqslant 1$, the function

$$F_{\gamma}(z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iz\tau} e^{-(\tau/\gamma^{\rho})^{2m}} d\tau, \quad z \in \mathbf{C},$$

is holomorphic in $\mathbb{C}$. Moreover, there exists four positive constants C_1, C_2, C_3 and C_4 (independent on γ) such that

$$|F_{\gamma}(z)| \leqslant C_1 \gamma^{\rho} e^{C_2 \gamma |\mathrm{Im} z|^{1/\rho}}, \quad \forall z \in \mathbb{C},$$

and

$$|F_{\gamma}(z)| \leqslant C_1 \gamma^{\rho} e^{-C_3 \gamma |\mathrm{Re} z|^{1/\rho}} \qquad \forall z \in \{z \in \mathbf{C},\ |\mathrm{Im} z| \leqslant C_4 |\mathrm{Re} z|\}$$

Weak observability for the Schrödinger eq.

Step 2: Connection between the Schrödinger's and the heat eqs.

Now, let s, $t \in \mathbb{R}$, we introduce the following Fourier-Bros-Iagolnitzer transformation as in (Lebeau-Robbiano)

$$w_{\gamma,t}(x,s) = \int_{\mathbb{R}} F_\gamma(t + is - \tau)\varphi(\tau)w(x,\tau)d\tau \ ,$$

where $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, $t \in I$ and $s \in (0,h)$.

Let v be a solution of the following boundary value problem in

$$\begin{cases} (i\partial_t + \Delta)v(x,t) = 0 & \text{in} \quad \Omega \times (0,T), \\ v = 0 & \text{on} \quad \partial\Omega \times (0,T). \end{cases}$$

In connection with the operator $(i\partial_t + \Delta)$, we define the parabolic operator in $\Omega \times (0,h)$ for some $h > 0$ by $(\partial_s + \Delta)$.
Since

$$(\partial_s + \Delta)v_{\gamma,t}(x,s) = \int_{\mathbb{R}} F_\gamma(t + is - \tau)(i\partial_\tau + \Delta)\left(\varphi(\tau)v(x,\tau)\right)d\tau$$

Connection between the Schrödinger's and heat eqs

We have $v_{\gamma,t}$ satisfies the following IBVP in $\Omega \times (0,h)$.

$$\begin{cases} (\partial_s + \Delta)\, v_{\gamma,t}(x,s) = G_{\gamma,t}(x,s) & \text{in} \quad \Omega \times (0,h), \\ v_{\gamma,t}(x,s) = 0 & \text{on} \quad \partial\Omega \times (0,h), \\ v_{\gamma,t}(x,0) = (F_\gamma * \varphi v(x,\cdot))(t) & \text{in} \quad \Omega, \end{cases}$$

where

$$G_{\gamma,t}(x,s) = -i \int_{\mathbb{R}} F_\gamma(r + is - \tau)\varphi'(\tau)u(x,\tau)d\tau.$$

Weak observability for the Schrödinger eq.

Step 3: Estimations

1. There extis $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ and $I \subset (0,T)$ such that

$$\|G_{\gamma,t}\|_{L^2(\Omega\times(0,h))} \leqslant Ce^{-C\gamma}\|v\|_{L^2(\Omega\times(0,T))}, \quad t \in I.$$

2.

$$\begin{aligned}\|v(x,\cdot)\|^2_{L^2(I)} &\leqslant \|\widehat{\varphi v}(x,\cdot) - \widehat{F}_\gamma \widehat{\varphi v}(x,\cdot)\|^2_{L^2(\mathbb{R})} + \int_I |v_{\gamma,t}(x,0)|^2\,dt \\ &\leqslant \frac{C}{\gamma^{2\mu}}\|\partial_t(\varphi v)(x,\cdot)\|^2_{L^2(\mathbb{R})} + \int_I |v_{\gamma,t}(x,0)|^2\,dt\end{aligned}$$

Thank you